Lie Extensions

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1. Introduction. In [8], Vessiot investigated the following system of ordinary differential equations

(1) $$\frac{dy_i}{dx} = \sum_{j=1}^{m} a_j X_j y_i \quad (1 \le i \le n),$$

which he called a "Lie system" after Lie's work [3]. Here the a_j denote functions in the independent variable x and X_j are linear differential operators in the shape

$$X_j = \sum_{i=1}^{n} \xi_{ji} \frac{\partial}{\partial y_i} \ (1 \le j \le m)$$

with the ξ_{ji} being functions of y, which constitute a Lie algebra over the field of complex numbers. Consideration of integrals of this turns out to be the same as of the differential operator

$$D = \frac{\partial}{\partial x} + \sum_{j=1}^{m} a_j X_j, \quad \left[\frac{\partial}{\partial x}, X_j\right] = 0,$$

which must satisfy

$$[D, X_i] = \sum_{j=1}^{m} \sum_{k=1}^{m} a_j c_{ijk} X_k,$$

with the c_{ijk} being the structure constants of the Lie algebra.

Here we shall examine the relationship between Lie systems and strongly normal extensions. To do that some preliminaries may be needed. Let K be an ordinary differential field of characteristic 0 with the differentiation D. In what follows we assume that the field of constants C_K of K is algebraically closed. Differential field extensions of K would be referred to be finitely generated as field extensions without notice. For a differential field extension R/K we adopt the usual notation $Der(R/K)$ for the Lie algebra consisting of all derivations of R over K. Differentiation D of R can be regarded as contained in $Der(R/C_K)$. Hence we can define the Lie product $[D, X]$ for $X \in Der(R/K)$, which is seen to lie therein. Let us denote by $\Omega^1(R/K)$ the dual R-vector space of $Der(R/K)$. It is generated with the differentials da of $a \in R$. Here $da(X) = Xa$ for $X \in Der(R/K)$. An additive endomorphism D of $\Omega^1(R/K)$ is defined by

$$(D\omega)X = D(\omega X) - \omega[D, X]$$
$$(\omega \in \Omega^1(R/K),\ X \in Der(R/K)).$$

Clearly $D(adb) = D(a)db + adDb$ holds for $a, b \in R$. Denote by $G(R/K)$ the group of all differential automorphisms of R/K. For every $\sigma \in G(R/K)$ we define two additive automorphisms σ_* and σ^* of $Der(R/K)$ and $\Omega^1(R/K)$ respectively by

$$\sigma_* X = \sigma X \sigma^{-1} \ (X \in Der(R/K)),$$
$$\sigma^* \omega = \sigma \omega \sigma_*^{-1} \ (\omega \in \Omega^1(R/K)).$$

Then $\sigma^*(adb) = \sigma(a) d\sigma b$ for $a, b \in R$.

Definition 1. We say that a differential field extension R/K is a *Lie extension* if $C_R = C_K$, there exists a C_K-Lie subalgebra $\mathfrak{g}$ of $Der(R/K)$ of finite dimension over C_K such that $[D, \mathfrak{g}] \subset K\mathfrak{g}$ and $R\mathfrak{g} = Der(R/K)$. In this case $\mathfrak{g}$ will be called its structure.

For instance we shall prove the following theorem:

Theorem 1. *Suppose that K is algebraically closed. Then every intermediate differential field of a strongly normal extension of K is a Lie extension of K.*

Recall that a differential field extension N/K is said to be strongly normal if $C_N = C_K$ and for every differential isomorphism σ

$$N\sigma N = NC_{N\sigma N} = \sigma NC_{N\sigma N}$$

holds. $G(N/K)$ turns out to be an algebraic group defined over C_K with the dimension equal to t.d. N/K. The structure of Lie extension N/K ocasionally can be constructed from invariant derivations, ones exchangeable with every differential automorphisms (cf. [2]).

In fact, strongly normal extensions are seen to be Lie extensions of the following special type.

Definition 2. A differential field extension R/K with $C_R = C_K$ is said to be *Lie closed* if $\Omega^1(R/K)$ possesses a basis of differentials which are annihilated by D. As seen in later, Lie closed extensions are Lie extensions.

A differential field extension R/K is said to depend rationally on arbitrary constants if there exists a differential field extension E/K such that E and R are free over K, and $ER = EC_{ER}$ holds. In this case R is contained in a strongly normal extension of K provided that $C_R = C_K$, and K is algebraically closed (cf. [5]). Such differential field extensions strongly relate to general solutions of algebraic differential equations with no movable singularities. Theorem 1 in particular tells us that any intermediate differential field of a Picard-Vessiot extension, a differential field extension generated with a fundamental solution of a system of linear homogeneous differential equations, and with no constant that is not contained in the coefficient field, is a Lie extension. Details for this are discussed in [3] and [8].

Theorem 2. *Let R/K be a Lie extension and $P_K(R)$ be the maximum among intermediate differential fields depending rationally on arbitrary constants. Then, $P_K(R)$ is also a Lie extension.*

It is worthwhile also to notice the statement in [7] that any solution of Lie system is one-valued, though it seems dubious. Although one might ask if one-valued functions satisfying algebraic differential equations are always obtained as solutions of Lie systems, it is seen incorrect because of the following fact.

Theorem 3. *The differential field generated by Painlevé's first transcendent over $C(x)$, the field of rational functions in x, $Dx = 1$, is not a Lie extension.*

It is very likely that Painlevé's first equation (2) admits no solution in a differential field extension of K which is Lie closed, provided it admits no solution algebraic over K.

2. Strongly normal extensions. Before proceeding to the proof of Theorem 1, we note the following.

Proposition 1. *Let R/K be a Lie extension with structure $\mathfrak{g}$. Suppose that an intermediate differential field S is stable under the action of $\mathfrak{g}$. Then S/K is also a Lie extension with the set $\mathfrak{g}' = \mathfrak{g}|_S$ of derivations $X|_S$ $(X \in \mathfrak{g})$ as Lie algebra structure.*

Proof. It is readily seen that Lie algebra $\mathfrak{g}'$ has structure constants in C_K. Let X be in $\mathfrak{g}'$ and Y be any extension of X to R. Then we have an expression $Y = \sum a_i X_i$, $a_i \in R$. Restricting this on S gives an expression for X with the use of restrictions of X_i on S. Regarding S as a linear subspace of K-linear space R, applying the projection of R onto S to the expression for X, we have the required expression for X with coefficients from S. This proves the proposition.

We fix a strongly normal extension N/K and denote by n its transcendence degree. Assume that K is algebraically closed. It is known that the set $g(N/K)$ of all invariant derivations is an n-dimensional K-linear space whose basis $X_1, X_2, \ldots, X_n$ also constitutes a basis for $Der(N/K)$. By definition $\sigma X_i = X_i \sigma$ for every $\sigma \in G(N/K)$ holds. A C_K-linear subspace $\mathfrak{g}$ generated with the X_i can be taken as a C_K-Lie algebra since $G(N/K)$ is defined over C_K and N is the function field of a principal homogeneous space for $G(N/K)$ over K. Clearly $[D, g(N/K)] \subset g(N/K)$, hence $[D, \mathfrak{g}] \subset K\mathfrak{g}$.

Proof of Theorem 1. Let R be an intermediate differential field and $\mathfrak{g}' = \mathfrak{g}|_R$. Since
$$\sigma X_i y = X_i \sigma y = X_i y$$
holds for $\sigma \in G(N/R)$ and $y \in R$, it follows that $X_i y \in R$, namely $\mathfrak{g}' R \subset R$. It is easily verified that $[D, \mathfrak{g}'] \subset K\mathfrak{g}'$. By Proposition 1, the proof is completed.

Proposition 2. *If the $\mathfrak{g}'$ in the proof just above satisfies $r = \dim_K K\mathfrak{g}' = t.d.R/K$, then the algebraic closure R^a of R in N is strongly normal over K.*

Proof. We may assume that restrictions $X_i|_R$ $(1 \leq i \leq r)$ constitute a basis for $Der(R/K)$. Inasmuch as each of other $X_j|_R$ depends linearly on $X_1|_R, \ldots, X_r|_R$ there are elements $a_{ji} \in K$ such that
$$Y_j = X_j - \sum_{i=1}^{r} a_{ji} X_i \in Der(N/R) \quad (r+1 \leq j \leq n).$$
The Y_j are contained in $g(N/K)$ as well, being linearly independent over N. For $\sigma \in G(N/K)$ and $y \in R$
$$Y_j \sigma y = \sigma Y_j y = 0.$$
Hence $\sigma y \in R^a$. This shows that R^a/K is strongly normal.

As the structure of the Lie extension N/K we can select particular one.

Proposition 3. *Suppose that K is algebraically closed. Then N/K is Lie closed, to say dually, there exists a Lie subalgebra $\mathfrak{g}$ of $Der(N/K)$ such that $N\mathfrak{g} = Der(N/K)$, $[D, \mathfrak{g}] = 0$.*

Proof. Let the ω_i be a basis of invariant differentials. Then they satisfy
$$D\omega_i = \sum_{j=1}^{n} a_{ij} \omega_j \quad (1 \leq i \leq n),$$

with $a_{ij} \in K$, because the left hand side is left invariant under the Galois group. We simply describe this as $D\omega = A\omega$, where ω denotes vector ${}^t(\omega_1, \dots, \omega_n)$ and $A = (a_{ij})$. Proposition asserts that matrix A can be reduced to 0 through suitable procedure. One of the properties of strongly normal extensions guarantees the existence of a differential field extension M/K which is differentially isomorphic to N/K and linearly disjoint from N over K, and satisfies $MN = NC_{MN} = MC_{MN}$. Let $c_i (1 \leq i \leq n)$ be a transcendence basis for C_{MN} over C_K and dc_i be the differentials in $\Omega^1(MN/M)$. Clearly $Ddc_i = 0$. We have the expression

$$\omega_i = \sum_{j=1}^{n} b_{ij} dc_j$$

with $b_{ij} \in MN$, $\det(b_{ij}) \neq 0$. Substitution of this into the equation satisfied by the ω_i leads to

$$Db_{ij} = \sum_{h=1}^{n} a_{ih} b_{hj}.$$

That is to say, $B = (b_{ij})$ is a fundamental matrix of the system of linear homogeneous differential equations, $DB = AB$. Since entries of B belong to NC_{MN}, there is a differential specialization over N

$$(c_1, \dots, c_n, B) \to (c'_1, \dots, c'_n, B')$$

such that $c'_i \in C_K$, $\det B' \neq 0$. B' is a matrix with entries from N satisfying $DB' = AB'$. Set $\eta = B'^{-1}\omega$. Then

$$D\eta = -B'^{-1}D(B')B'^{-1}\omega + B'^{-1}D\omega$$
$$= -B'^{-1}A\omega + B'^{-1}A\omega = 0.$$

3. Lie closedness. We show that if R/K is Lie closed with basis ω_i for $\Omega^1(R/K)$ such that $D\omega_i = 0$, then

$$d\omega_i = -\frac{1}{2}\sum_{j,k} e_{ijk}\omega_j \wedge \omega_k \qquad (e_{ijk} \in C_K,\ e_{ijk} = -e_{ikj}).$$

In fact

$$0 = Dd\omega_i = -\frac{1}{2}\sum_{j,k} D(e_{ijk})\omega_j \wedge \omega_k,$$

therefore $e_{ijk} \in C_R = C_K$. Here we used the readily understandable extension of D in $\Omega^2(R/K)$, which satisfies

$$D(\alpha \wedge \beta) = D(\alpha) \wedge \beta + \alpha \wedge D(\beta)$$

for $\alpha, \beta \in \Omega^1(R/K)$. The dual basis X_i in $Der(R/K)$ to the ω_i constitutes a Lie algebra with structure constants in C_K, hence turns R/K into a Lie extension.

Particular type of strong normality makes easier construction of such Lie algebras mentioned in Proposition 3. If the Galois group of N/K is an abelian variety, then the fundamental matrix utilized in the preceding section must be rational over K because the Galois group of $K(\Phi)/K$, Φ being rational over N as proved in the preceding section, is affine, hence 0-dimensional. If N/K is a Picard-Vessiot extension generated with fundamental matrix Φ satisfying linear differential equation $D\Phi = A\Phi$ over K, in $\Omega^1(N/K)$ we define

$$\omega = \Phi^{-1}d\Phi,$$

where ω indicates a matrix with entries in $\Omega^1(N/K)$. Clearly $D\omega = 0$.

The same method applies Lie closed extensions in a little more general situation.

Proposition 4. *Let R/K be a differential field extension with $C_R = C_K$ and n denote its transcendence degree. Suppose that there is an n-dimensional K-linear subspace V of $Der(R/K)$ such that $RV = Der(R/K)$ and $[D, V] \subset V$. Then there exists a differential field extension R^c/K being Lie closed which includes R.*

Proof. If we take a basis X_i for V, then

$$[D, X_i] = \sum_{h=1}^{n} a_{ih}X_h, \quad a_{ih} \in K.$$

Let R^c be a Picard-Vessiot extension of R generated with fundamental matrix Φ of

$$D\Phi = A\Phi, \quad A = (a_{ih}).$$

Noting $Dd\Phi = Ad\Phi$, we have

$$[D, \Phi^{-1}X] = 0, \quad D(\Phi^{-1}d\Phi) = 0.$$

The first implies the existence of a basis for $R^c\Omega^1(R/K)$, each differential in which vanishes through D. From this basis together with some of differentials given from $\Phi^{-1}d\Phi$ we can construct a basis for $\Omega_{R^c/K}$, each differential in which D annihilates.

4. Proof of Theorem 2. Any differential field extension R/K has a unique intermediate differential field $P_K(R)$ which is the maximum among intermediate differential fields depending rationally on arbitrary constants (cf. [5]).

Proposition 5. *If there is a finite dimensional K-linear subspace V of $Der(R/K)$ with $[D, V] \subset V$, then $VP_K(R) \subset P_K(R)$.*

Proof. By [5] there is a differential field extension E/K such that R, E are free over K and $P_E(ER) = EP_K(R) = EC_{ER}$ holds. Every derivation in V can be extended to the derivation in $Der(ER/E)$, for which we will exploit the same symbol. Let c be a constant of ER. Let the X_i be a basis for V. Then we have the linear combination with coefficients from $K \subset E$

$$[D, X_i] = \sum_h a_{ih} X_h.$$

Applying these onto c we get

$$DX_i c = \sum_h a_{ih} X_h c.$$

This indicates that the set of elements $X_i c$ is a solution of a system of linear differential equations defined over E. Hence the $X_i c$ are contained in $P_E(ER)$, therefore $VP_E(ER) \subset P_E(ER)$, namely $V(EP_K(R)) \subset EP_K(R)$. Applying the projection of $K^a E$ onto K^a to this inclusion, we conclude that $VP_K(R) \subset P_K(R)$, noting $P_K(R) \supset K^a$, K^a denoting the algebraic closure of K in R.

As a corollary we obtain that if R/K is a Lie extension then so is $P_K(R)/K$ using Proposition 1, though the same result is drawn from the fact that $P_K(R)$ is contained in a strongly normal extension provided K is algebraically closed.

5. Painlevé's first transcendent. Here is given a proof of Theorem 3. Painlevé's first transcendent is defined to be a solution of

(2) $$D^2 y = 6y^2 + x, \quad Dx = 1$$

over $\boldsymbol{C}(x)$. We shall prove that if K contains x with $C_K = \boldsymbol{C}$ and $R = K\langle y \rangle / K$ is a Lie extension then there exists a solution of (2) being algebraic over K, which implies the theorem since (2) admits no algebraic function as solution. By Proposition 4 there exists a fundamental matrix Φ of a system of linear differential equations defined over K such that $R(\Phi)/K$ is Lie closed. Suppose equation (2) has no solution algebraic over $K(\Phi)$. Then R and $K(\Phi)$ are freee over K, and so that $\Omega^1(R(\Phi)/K(\Phi))$ is generated with $\Omega^1(R/K)$. The Picard-Vessiot group attached to $R(\Phi)/K(\Phi)$ agrees precisely with $SL_2(\boldsymbol{C})$ according to [6]. Incidentally $R(\Phi)/K(\Phi)$ is Lie closed, since so is $R(\Phi)/K$. This is absurd, showing that equation (2) admits a solution algebraic over $K(\Phi)$, hence so does over K, on account of the property of (2) which is proved in [4].

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