On the Asymptotic Behavior of the Occupation Time in Cones of d-dimensional Brownian Motion

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1. Introduction. The purpose of this article is to prove some new feature on the occupation time of Brownian motion. Let $\{B_u\}$ be a d-dimensional Brownian motion starting from the origin and C be a cone with the vertex at the origin. We denote the occupation time of B_u in C by $A_t = \int_0^t 1_C(B_u)\,du$.

In the only case of $d = 1$ and $C = [0, \infty)$, the explicit law of A_1 is known as Lévy's first arcsin law:

$$P(A_1 \leq s) = \frac{2}{\pi} \arcsin \sqrt{s} \text{ for } 0 \leq s \leq 1.$$

In higher demensions, however, any explicit law of A_1 is not known. We shall present a partial result giving certain asymptotic explicitly. Note that in the one-dimensional case,

$$s^{-\frac{1}{2}} P(A_1 \leq s) \to \frac{2}{\pi} \text{ as } s \to 0.$$

The main result of this article is the following generalization:

Theorem. *Let each of $\{C_i\}_{1\leq i\leq N}$ be a closed cone with the vertex at the origin and simply connected on S^{d-1} and have C^2-class boundary except the origin. Suppose that the cone C is expressed as $C = \cup_{1\leq i\leq N} C_i$ and satisfies that $C^c \cap S^{d-1}$ is connected on S^{d-1}. Then there exists a positive constant k such that*

$$s^{-\zeta} P(A_1 \leq s) \to k \text{ as } s \to 0,$$

where ζ is defined by $2\zeta = \sqrt{\left(\frac{d}{2} - 1\right)^2 + 2\lambda_1} - \left(\frac{d}{2} - 1\right)$ and λ_1 denotes the first eigenvalue of $-\Delta/2$ on $C^c \cap S^{d-1}$ ($S^{d-1} = \{x \in \boldsymbol{R}^d; |x| = 1\}$) with Dirichlet boundary condition.

Remark. We should note that in order to make $C^c \cap S^{d-1}$ connected, we need to set $N = 1$ when $d = 2$. In particular, $\zeta = \pi/(2(2\pi - \theta))$ when $d = 2$, where θ denotes the angle of the cone around the origin.

We would like to remark also the following corollary immediately obtained by considering the relation of $\int_0^1 1_{C^c}(B_u)\,du = 1 - \int_0^1 1_C(B_u)\,du$.

Corollary. *Suppose that the cone C^c satisfies the same conditions as in the above theorem. Then there exists a positive constant k' such that*

$$(1 - s)^{-\zeta'} P(A_1 \geq s) \to k' \text{ as } s \to 1,$$

where ζ' is defined by λ_1' similarly as in the theorem, and λ_1' denotes the first eigenvalue of $-\Delta/2$ on $C \cap S^{d-1}$ with Dirichlet boundary condition.

There have been some efforts to get its asymptotic behavior. T. Meyre and W. Werner [5] proved that if the cone C is convex, there exist two constants k_1, k_2 depending only on C such that

$$k_1 s^{\zeta} \leq P(A_1 \leq s) \leq k_2 s^{\zeta} \tag{1.1}$$

holds for all $0 \leq s \leq 1$. Recently R. Bass and K. Burdzy [1] proved that if C is a closed cone which satisfies that $C^c \cap S^{d-1}$ is connected and $\nu(\partial C) = 0$ (ν denotes the Lebesgue measure on $\boldsymbol{R}^d$), then we have

$$\lim_{s\to 0} \frac{\log P(A_1 \leq s)}{\log s} = \zeta. \tag{1.2}$$

Both of the above results rely heavily on the estimate of the hitting time to the cone. Since the behavior of the hitting time plays an important role in our proof, we devote Section 2 to investigating it, and in Section 3 we shall give the proof of the main result.

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2. Estimate of the hitting time. In oder to prove our theorem we need a more precise estimate of the hitting time than that employed in the proofs of (1.1) and (1.2). The estimate in Proposition 1 is essential in our proof.

We set $\sigma = \inf\{t \geq 0; B_t \in \partial C\}$. Let $\phi(x, y)$ be the angle not exceeding π between ℓ_x and ℓ_y where $\ell_x(\ell_y)$ denotes the line connecting $x(y)$ and the origin. In the case where A is a subset in

$\boldsymbol{R}^d \setminus \{O\}$, we define $\phi(x, A) = \inf\{\phi(x, y); y \in A\}$.

Proposition 1. *For any fixed constant $t_0 > 1$, there exists a constant K_1 depending only on the cone C and t_0 such that*

$$(2.1) \qquad P_z(\sigma > t) \leq K_1 \left(\frac{|z|^2}{t}\right)^{\zeta} \phi(z, C)$$

for any $t > 0$, $z \in C^c$ satisfying $t \geq t_0 |z|^2$ with ζ as in Theorem.

Remark. The estimate which is used in the proofs of (1.1) and (1.2) do not have the term $\phi(z, C)$. The term $\phi(z, C)$ plays an essential role for our purpose.

Proof of Proposition 1. It suffices to prove (2.1) for $z \in C^c \cap S^{d-1}$, $t \geq t_0$ thanks to the scaling properties of Brownian motion. Let $\{\lambda_n\}$ be a sequence of eigenvalues of $-\Delta/2$ on $C^c \cap S^{d-1}$ with Dirichlet boundary condition, and $\{\varphi_n\}$ be a sequence of eigenfunctions corresponding to $\{\lambda_n\}$, which forms a complete orthonormal basis of $L^2(C^c \cap S^{d-1}, d\mu)$. Here $d\mu$ denotes the Lebesgue measure on S^{d-1} normalized to have total mass $2\pi^{d/2}/\Gamma(d/2)$. For $a \in \boldsymbol{R}^d \setminus \{O\}$, let $\{R_t\}$ be a Bessel process starting from $|a|$ and $\{\theta_t\}$ be a Brownian motion on S^{d-1} starting from $a/|a|$ which is independent of $\{R_t\}$. We can represent a d-dimensinal Brownian motion B_t starting from a as

$$B_t = R_t \theta_{U_t}, \text{ where } U_t = \int_0^t \frac{ds}{R_s^2}.$$

We set $\tilde{\sigma} = \inf\{t \geq 0; \theta_t \in C \cap S^{d-1}\}$. By the relation of $\tilde{\sigma} = U_\sigma$ and independence of $\{R_t\}$ and $\{\theta_t\}$, we have for $Z \in C^c \cap S^{d-1}$

$$\begin{aligned} P_Z(\sigma > t) &= P_z(U_\sigma > U_t) \\ &= \int_0^\infty P_z(\tilde{\sigma} > s) P_z(U_t \in ds) \\ &= \sum_{n=1}^\infty \varphi_n(z) \int_{C^c \cap S^{d-1}} \varphi_n d\mu \, E_z(\exp(-\lambda_n U_t)). \end{aligned}$$

Applying Yor's result [9] to the term $E_z(\exp(-\lambda_n U_t))$, we obtain through some calculation that for some constant L_1

$$(2.2) \quad P_z(\sigma > t) \leq L_1 t^{\frac{1}{2}(\frac{d}{2}-1)} \sum_{n=1}^\infty t^{-\frac{1}{2}\sqrt{(\frac{d}{2}-1)^2 + 2\lambda_n}} |\varphi_n(z)|.$$

Now we have to estimate the terms $|\varphi_n(z)|$. Let $q_C(t, x, y)$ be a transition density for Brownian motion on $C^c \cap S^{d-1}$ killed when it hits C. Set $G_C(x, y) = \int_0^\infty q_C(t, x, y) dt$. Using DeBlassie's results [3] on the eigenfunctions on S^{d-1}, we see that for some constants L_2 and L_3

$$\begin{aligned} (2.3) \quad |\varphi_n(z)| &\leq \lambda_n \int_{C^c \cap S^{d-1}} G_C(z, x) |\varphi_n(x)| \mu(dx) \\ &= \lambda_n E_z\left(\int_0^{\tilde{\sigma}} |\varphi_n(\theta_s)| ds\right) \\ &\leq L_2 \lambda_n^{L_3} E_z \tilde{\sigma}. \end{aligned}$$

On the other hand, we can see by little thought that there exists a constant L_4 such that for any $z \in C^c \cap S^{d-1}$

$$E_z \tilde{\sigma} \leq L_4 \phi(z, C).$$

This together with (2.2) and (2.3) completes the proof of Proposition 1.

In order to prove our theorem, we need the following two lemmas.

Lemma 2. *There exists a domain D satisfying the following three conditions:*

(i) $\partial C \subset D$.

(ii) $0 < \inf_{x \in \partial C} d(x, D^c)$.

(iii) *Let τ be the first exit time from D, i.e. $\tau = \inf\{t \geq 0; B_t \notin D\}$. Then there are two positive constants K_2, K_3 such that $P_z(\tau > t) \leq K_2 e^{-K_3 t}$ for any $z \in D$, $t \geq 0$.*

We set $r_0 = \inf_{x \in \partial C} d(x, D^c)$ and redefine $D = \cup_{x \in \partial C}\{y \in \boldsymbol{R}^d; |y - x| < r_0\}$, then D clearly satisfies the condition (i), (ii), and (iii). Now we set $D_1 = \partial D \cap C$ and $D_2 = \partial D \cap C^c$.

Lemma 3. *There exist two positive constants K_4, K_5 such that for any $z \in D_2$, $t \geq 0$*

$$t^\zeta P_z(\sigma > t) \leq \begin{cases} K_4 & \text{if } 0 < \zeta \leq \dfrac{1}{4} \\ K_5 |z|^{2\zeta - \frac{1}{2}} & \text{if } \zeta > \dfrac{1}{4} \end{cases}.$$

3. Proof of the theorem. Let σ, τ, D_1 and D_2 be as before.

We show that there exists a positive constant M determined by the cone C such that

$$(3.1) \qquad E_0(e^{-A_t}) \sim \frac{M}{t^\zeta} \text{ as } t \to \infty.$$

If we obtain (3.1), we can prove our theorem easily by applying Tauberian theorem to the identity

$$E_0(e^{-A_t}) = \int_0^\infty e^{-ts} dP_0(A_1 \leq s).$$

We mention briefly the idea to prove (3.1).

Define sequences of stopping times inductively as follows:

$$\begin{cases} \sigma_0 = 0 \\ \tau_n = \sigma_n + \tau \circ \theta_{\sigma_n} & (n = 0,1,2,\ldots) \\ \sigma_n = \tau_{n-1} + \sigma \circ \theta_{\tau_{n-1}} & (n = 1,2,3,\ldots), \end{cases}$$

where $\theta.$ is the path shift operator. Now we have

$$(3.2)\quad t^{\zeta}E_0(e^{-A_t}) = \sum_{n=1}^{\infty} t^{\zeta}E_0(e^{-A_t}; \sigma_n \leq t < \sigma_{n+1}).$$

Applying the strong Markov property to the n-th term of (3.2), we have

$$(3.3)\ E_0\Big(e^{-A_{\sigma_n}}1_{(\sigma_n\leq t)}\Big(\frac{t}{a}\Big)^{\zeta}a^{\zeta}E_{B_{\sigma_n}}(e^{-A_a}; a<\tau)|_{a=t-\sigma_n}\Big)$$
$$+ E_0\Big(e^{-A_{\tau_n}}1_{(\tau_n\leq t)}\Big(\frac{t}{b}\Big)^{\zeta}b^{\zeta}E_{B_{\tau_n}}(e^{-b}; b<\sigma)\,|_{b=t-\tau_n};\ B_{\tau_n}\in D_1\Big)$$
$$+ E_0\Big(e^{-A_{\tau_n}}1_{(\tau_n\leq t)}\Big(\frac{t}{b}\Big)^{\zeta}b^{\zeta}P_{B_{\tau_n}}(b<\sigma)\,|_{b=t-\tau_n};\ B_{\tau_n}\in D_2\Big).$$

We see that the first term is negligible by Lemma 2 when t is large enough. Moreover, because of the term e^{-b}, we see also that the second term is negligible. To establish the convergence of the last term, Lemma 3 is employed crucially to show the uniform integrability of it. Applying the following formula which has been obtained by M. Shimura [8] for the case $d=2$ and by R. D. DeBlassie [3] for the case $d \geq 3$: for any $z \in C^c$

$$t^{\zeta}P_z(\sigma > t) \rightarrow \frac{\Gamma\Big(\zeta+\frac{d}{2}\Big)}{\Gamma\Big(2\zeta+\frac{d}{2}\Big)}\int_{C^c\cap S^{d-1}}\varphi_1 d\mu\, \varphi_1\Big(\frac{z}{|z|}\Big)\Big(\frac{|z|^2}{2}\Big)^{\zeta} \quad as\ t\rightarrow\infty,$$

we see that the last term of (3.3) converges to

$$(3.4)\quad \frac{\Gamma\Big(\zeta+\frac{d}{2}\Big)}{\Gamma\Big(2\zeta+\frac{d}{2}\Big)}\int_{C^c\cap S^{d-1}}\varphi_1 d\mu$$
$$E_0\Big(e^{-A_{\tau_n}}\varphi_1\Big(\frac{B_{\tau_n}}{|B_{\tau_n}|}\Big)\Big(\frac{|B_{\tau_n}|^2}{2}\Big)^{\zeta};\ B_{\tau_n}\in D_2\Big).$$

As a result, we see that the left term of (3.2) converges to the sum of (3.4).

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