A Form of Classical Liouville Theorem

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The Liouville theorem in the theory of harmonic functions (cf. e.g. Axler *et al.* [1]) states that any nonnegative harmonic function u on the d-dimensional Euclidean space $\boldsymbol{R}^d$ $(d \geqq 2)$ reduces to a constant. It naturally occurs the question how much the condition for u to be nonnegative can be relaxed (see, e.g. Doob [3]). Recently Bourdon [2] proposed, among other related things, the following interesting generalization of the Liouville theorem:

Theorem A (Liouville Theorem). *If u is a harmonic function on $\boldsymbol{R}^d$ and satisfies*

$$\liminf_{|x|\to\infty} \frac{u(x)}{|x|} \geqq 0, \tag{1}$$

then u is a constant function on $\boldsymbol{R}^d$.

Bourdon gave an elementary and simple proof to the above result by using only the mean value property of harmonic functions originally due to an ingenious idea of Nelson [6](cf. also [1]). In contrast with the Liouville theorem in the theory of complex functions it is natural to consider Theorem A as a special case of the following result:

Theorem B (Liouville Theorem). *If u is a harmonic function on $\boldsymbol{R}^d$ and satisfies*

$$\liminf_{|x|\to\infty} \frac{u(x)}{|x|^{n+1}} \geqq 0 \tag{2}$$

for some nonnegative integer n, then u is a harmonic polynomial on $\boldsymbol{R}^d$ of degree at most n.

Clearly the $n = 0$ case of Theorem B is nothing but Theorem A. The Bourdon proof of Theorem A seems not to be straightforwardly applied to that for Theorem B. It has been constantly our claim (cf. e.g. [4]) that the Fourier expansion method is one of the best tools to handle harmonic functions as far as their domains of definition are rotationally invariant such as $\boldsymbol{R}^d$. The purpose of this note is to give a proof to Theorem B by using the Fourier expansion, and actually, we prove Theorem B in the following superficially more general form:

3. Theorem (Liouville Theorem). *Suppose that u is a harmonic function on $\boldsymbol{R}^d$ and that there exists an increasing divergent sequence $(r_m)_{m\geqq 1}$ of positive numbers r_m such that*

$$\liminf_{m\to\infty} \left(\min_{|x|=r_m} \frac{u(x)}{|x|^{n+1}} \right) \geqq 0 \tag{4}$$

for some nonnegative integer n, then u is a harmonic polynomial on $\boldsymbol{R}^d$ of degree at most n.

Proof. We use the polar coordinate $x = r\xi$ for points $x \in \boldsymbol{R}^d$, where $r = |x| \geqq 0$ and $\xi = x/|x| \in S^{d-1}$ for $x \neq 0$ and $\xi = (1, 0, \ldots, 0) \in S^{d-1}$ for $x = 0$ for definitness. Here S^{d-1} is the unit sphere $\{x \in \boldsymbol{R}^d : |x| = 1\}$. We choose and then fix an orthonormal basis $\{S_{kj} : j = 1, \ldots, N(k)\}$ of the subspace of all spherical harmonics of degree k of $L^2(S^{d-1}, d\sigma)$, where $d\sigma$ is the area element on S^{d-1}. Then $\{S_{kj} : j = 1, \ldots, N(k)\,;\, k = 0, 1, \ldots\}$ is a complete orthonormal system in $L^2(S^{d-1}, d\sigma)$. We have, as the special case of the addition theorem,

$$\sum_{j=1}^{N(k)} S_{kj}(\xi)^2 = \frac{N(k)}{\sigma_d},$$

where σ_d is the surface area $\sigma(S^{d-1})$ of S^{d-1}. Here $N(0) = 1$ and $N(k) = (2k + d - 2)\Gamma(k + d - 2)/\Gamma(k+1)\Gamma(d-1)$ for $k = 1, 2, \ldots$. For simplicity we set $A_k := \sqrt{N(k)/\sigma_d}$ so that $|S_{kj}(\xi)| \leqq A_k$ $(j = 1, \ldots, N(k)\,;\, k = 0, 1, \ldots)$. Then we have the following expansion of $u(r\xi)$ in terms of spherical harmonics $\{S_{kj}\}$:

$$u(r\xi) = \sum_{k=0}^{\infty} \left(\sum_{j=1}^{N(k)} a_{kj} S_{kj}(\xi) \right) r^k, \tag{5}$$

where a_{kj} $(j = 1, \ldots, N(k)\,;\, k = 0, 1, \ldots)$ are

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constants. Here the series on the right hand side of (5) converges uniformly in $\xi \in S^{d-1}$ for any fixed $0 < r < \infty$.

The condition (4) assures that, for any positive number $\varepsilon > 0$, there exists a number m_0 such that

$$u(r_m\xi)/r_m^{n+1} \geqq -\varepsilon$$

for every $m \geqq m_0$. This means that

$$(6) \qquad \sum_{k=0}^{\infty}\left(\sum_{j=1}^{N(k)} a_{kj}S_{kj}(\xi)\right) r_m^k + \varepsilon r_m^{n+1} \geqq 0$$

for all $\xi \in S^{d-1}$ and for all $m \geqq m_0$. Multiply $A_k \pm S_{kj}(\xi) \geqq 0$ to both sides of (6) and then integrate both sides of the resulting inequality over S^{d-1} with respect to $d\sigma$. (Present authors have been using this device frequently (see e.g. [4], [5], etc.)). Then we obtain

$$\sigma_d A_k(a_{01}/\sqrt{\sigma_d} + \varepsilon r_m^{n+1}) \pm a_{kj} r_m^k \geqq 0,$$

or equivalently, we have

$$(7) \quad \sigma_d A_k(a_{01} r_m^{-k}/\sqrt{\sigma_d} + \varepsilon r_m^{n+1-k}) \pm a_{kj} \geqq 0$$

for every $k \geqq 1$ and every $m \geqq m_0$. If $k > n+1$, then on letting $m \uparrow \infty$ in (7) we deduce $\pm a_{kj} \geqq 0$ so that $a_{kj} = 0$ $(j = 1, \ldots, N(k)\,;\ k \geqq n+2)$. If $k = n+1$, then again on letting $m \uparrow \infty$ in (7), we see that

$$\sigma_d A_{n+1}\varepsilon \pm a_{n+1,j} \geqq 0.$$

Here ε may be any positive number and thus on making $\varepsilon \downarrow 0$ in the above, we conclude that $\pm a_{n+1,j} \geqq 0$ or $a_{n+1,j} = 0$ $(j = 1, \ldots, N(n+1))$. Therefore (5) is reduced to

$$(8) \qquad u(r\xi) = \sum_{k=0}^{n}\left(\sum_{j=1}^{N(k)} a_{kj}S_{kj}(\xi)\right) r^k.$$

Since $S_{kj}(x) := r^k S_{kj}(\xi)$ $(x = r\xi)$ is a homogeneous harmonic polynomial in x of degree k, (8) yields that $u(x) = u(r\xi)$ is a harmonic polynomial of degree at most n. □

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