Asymptotic Behavior of Solutions of Certain Second Order Differential Equations at Infinity

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In his work [3], the author encountered differential equations which can be reduced to the following form:

(1) $$u''(t) + f(t)u(t) = 0$$

Here, u is an unknown function and f is a given coefficient function. In this paper, the order of decay of $u(t)$ as $t \to \infty$ is explicitly given in terms of $f(t)$.

To describe the results, we introduce a function space. For two positive numbers p and A, let $\mathcal{F}_{a,p,A}$ be the totality of real-valued C^2-functions f on an open interval containing $I = [a, \infty)$, $a \in \boldsymbol{R}$, satisfying the following conditions:

(i) $f(t)$ is positive and convex on I;

(ii) the inequality $f(t)f''(t) \leq (p+1)f'(t)^2$ holds for $t \in I$;

(iii) for $t \in I$, $f'(t) \geq Af(t)$.

The space of all real-valued solutions u of (1) is denoted by $\mathcal{S}_f$. The main result of this paper is the following theorem.

Theorem. *If the coefficient function f in* (1) *is in the family $\mathcal{F}_{a,p,A}$ for some $0 < p < \frac{1}{4}$, $a \in \boldsymbol{R}$ and $A > 0$. Then $u(t) = O(f(t)^{-1/4})$ as $t \to \infty$ for all u in $\mathcal{S}_f$.*

Moreover, an upper bound for $|u(t)f(t)^{1/4}|$ can be explicitly given (see Proposition 7).

1. Properties about zeros of a solution of equation (1). As a first step of our proof, we give an estimate for solutions of (1).

Lemma 1. *Let f be a function in $\mathcal{F}_{a,p,A}$ with $0 < p \leq \frac{1}{4}$ and u a solution of* (1). *Assume that for two real numbers α and β $(a \leq \alpha < \beta)$, $u(\alpha) = 0$ and $u(t) \neq 0$ for $\alpha < t < \beta$. Then it holds that*

$$0 \leq u'(\alpha)^{-1}u(t) \leq f(\alpha)^{-1/4}f(t)^{-1/4} \sin\left(\int_\alpha^t f(s)^{1/2}ds\right)$$

for $\alpha \leq t \leq \beta$.

The following is an immediate consequence of the definition of the family $\mathcal{F}_{a,p,A}$.

Lemma 2. (*i*) *If $0 < p_1 < p_2$, then $\mathcal{F}_{a,p_1,A} \subset \mathcal{F}_{a,p_2,A}$ for all $a \in \boldsymbol{R}$, $A > 0$.*

(ii) *Let f be a function in $\mathcal{F}_{a,p,A}$, then $f'(t) \leq C \cdot f(t)^{p+1}$ for $t \geq a$, where $C = f'(a)f(a)^{-(p+1)}$*

Let f be a function in $\mathcal{F}_{a,p,A}$ and u in $\mathcal{S}_f$. Then, it is easily seen that u has infinitely many zeros on the interval $[a, \infty)$. Since the set of all zeros of u has no accumulation point, we enumerate the totality of zeros of u in $[a, \infty)$ in an increasing order by $\{\alpha_n \mid n = 1, 2, \dots\}$. Then we have the following relations between the growth of $f(t)$ and the distribution of zeros of u.

Lemma 3. *Let f and u be functions in $\mathcal{F}_{a,1/4,A}$ and in $\mathcal{S}_f$ respectively. Take a constant C in such a way that $f'(t) \leq C \cdot f(t)^{5/4}$ $(\forall\, t \geq \alpha_1)$.* (*Such a constant C always exists by Lemma* 2.) *Assume that $k > 1$ be a constant and that $f(t) \geq 25C^4[256(k^2 - 1)^2]^{-1}$ for $\geq t_0 (> \alpha_1)$. Then $\int_{\beta_1}^{\beta_2} f(t)^{1/2}dt \geq k^{-1}\pi$ for any two adjacent zeros β_1 and β_2 of u such that $t_0 \leq \beta_1 < \beta_2$.*

Lemma 4. *Let f, u and $\{\alpha_n\}$ be as above. Then,*

$$\int_{\alpha_n}^{\alpha_{n+1}} f(t)^{1/2}dt \leq \pi \ (\forall\, n) \text{ and}$$

$$\lim_{n\to\infty} \int_{\alpha_n}^{\alpha_{n+1}} f(t)^{1/2}dt = \pi.$$

Let $f \in \mathcal{F}_{a,p,A}$ $(0 < p < \frac{1}{4})$ and $u \in \mathcal{S}_f$. Since $\mathcal{F}_{a,p,A} \subset \mathcal{F}_{a,1/4,A}$, we can take a constant $T > a$ such that $f(t)^6 f'(t)^{-4} \geq 1/40$ for $t \geq T$. Since we are considering the behavior of functions in $\mathcal{S}_f$ as $t \to \infty$, we may remove finitely many terms in $\{\alpha_n\}$ and renumber them in such a way that $\alpha_1 \geq T$.

Now, define constants C_0, j_f and t_j $(j = 0, 1, \dots)$ by

$$C_0 = f'(T)f(T)^{-5/4},$$
$$j_f = \lfloor 2\sqrt{10}\, C_0^{-2} f(\alpha_1)^{1/2} \rfloor,$$
$$f(t_0) = \max\{f(T),\ C_0^4 j_f^2/40\},$$

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$$f(t_j) = C_0^4(j+j_f)^2/40,$$

for $j > 0$. Here, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.

Note that $j_f > 0$ and that t_j's are uniquely determined. It is easily seen that $t_j < t_{j+1}$ for all j and $\lim_{j\to\infty} t_j = \infty$. We define sets of integers I_j by $I_j = \{n \mid t_j \leq \alpha_n < t_{j+1}\}$.

Lemma 5. *For a function f in $\mathscr{F}_{a,p,A}$ $(0 < p < \frac{1}{4})$ and $u \in \mathscr{S}_f$, define I as above. Then, the number of elements in I_j is majorized by a constant $N = 1 + \lfloor \sqrt{30}\, C_0^2 [5A\pi]^{-1} \rfloor$, independent of $u \in \mathscr{S}_f$, for all $j = 0, 1, \ldots$.*

2. Sketch of a proof of the main result. Lemmas 1 to 5 altogether prove Lemma 6 below. Then Lemmas 1 and 6 give our theorem.

Lemma 6. *Let f, u, T and $\{\alpha_n\}$ be as mentioned before the previous lemma. Then the sequence $\{|u'(\alpha_n)| f(\alpha_n)^{-1/4}\}$ is bounded.*

Sketch of proof. By Lemma 2(ii), we can take two constants C_0 and C_1 so that $f'(t) \leq C_0 \cdot f(t)^{3/4}$ and $f'(t) \leq C_1 \cdot f(t)^{p+1}$ for $t \geq T$. Also by Lemma 4, $\int_{\alpha_n}^{\alpha_{n+1}} f(s)^{1/2} ds \leq \pi$ for $n = 1, 2, \ldots$. Define j_f, t_j $(j = 0, 1, \ldots)$ and I_j as mentioned before Lemma 5.

Since α_n and α_{n+1} are adjacent zeros of u, the signatures of $u'(\alpha_n)$ and $u'(\alpha_{n+1})$ are mutually distinct. If $n \in I_j$, then we obtain the following inequality by using Lemma 1 and partial integrations. In this computation, $F(t)$ stands for $\int_{\alpha_n}^{t} f(s)^{1/2} ds$.

$$\begin{aligned}
0 &> u'(\alpha_n)^{-1} u'(\alpha_{n+1}) \\
&= u'(\alpha_n)^{-1}\left[u'(\alpha_n) + \int_{\alpha_n}^{\alpha_{n+1}} u''(t)\,dt\right] \\
&\geq 1 - f(\alpha_n)^{-1/4} \int_{\alpha_n}^{\alpha_{n+1}} f(t)^{1/4} \cdot f(t)^{1/2} \sin F(t)\,dt \\
&\geq -f(\alpha_n)^{1/4} f(\alpha_{n+1})^{1/4} - \frac{1}{4} C_0 f(\alpha_n)^{-1/4} \sin F(\alpha_{n+1}) \\
&\quad - \frac{5}{16} C_1^2 f(\alpha_n)^{-1/4} \int_{\alpha_n}^{\alpha_{n+1}} f(t)^{(8p-3)/4} \cdot f(t)^{1/2} \sin F(t)\,dt \\
&\geq -f(\alpha_n)^{1/4} f(\alpha_{n+1})^{1/4} - \frac{1}{4} C_0 f(\alpha_n)^{-1/4} \sin F(\alpha_{n+1}) \\
&\quad - \frac{5}{8} C_1^2 f(\alpha_n)^{2p-1}.
\end{aligned}$$

Put $r_n = u'(\alpha_n) f(\alpha_n)^{-1/4}$, then the above inequalities imply that

$$(2)\quad |r_{n+1} r_n^{-1}| \leq 1 + \frac{1}{4} f(\alpha_{n+1})^{-1/4} C_0 \sin F(\alpha_{n+1}) + \frac{5}{8} C_1^2 f(\alpha_n)^{(8p-3)/4} f(\alpha_{n+1})^{-1/4}.$$

Since $f(t_j) > 25 C_0^2 \left\{256 \left[(1 + (j+j_f)^{-1})^2 - 1\right]^2\right\}^{-1}$, Lemma 3 gives that

$$\pi \geq F(\alpha_{n+1}) \geq \pi[1 + (j+j_f)^{-1}]^{-1} > \pi(1 - (j+j_f)^{-1}).$$

Combined with this estimate, the inequality (2) gives that if $n \in I_j$,

$$\begin{aligned}
|r_{n+1} r_n^{-1}| &\leq 1 + \frac{1}{4} f(t_j)^{-1/4} C_0 \cdot \pi (j+j_f)^{-1} \\
&\qquad + \frac{5}{8} C_1^2 f(t_j)^{2p-1} \\
&\leq 1 + \frac{1}{4} \sqrt[4]{40}\, \pi (j+j_f)^{-\frac{3}{2}} \\
&\qquad + C_2 (j+j_f)^{4p-2},
\end{aligned}$$

for a positive constant C_2 depending only on f.

Lemma 5 gives a constant N independent of each u in $\mathscr{S}_f$ such that $|I_j| \leq N$ for all j. So, $|r_n r_2^{-1}|$ $(n \in I_j)$ is majorized by a constant as in the following way:

$$\begin{aligned}
|r_n r_2^{-1}| &\leq \prod_{i=2}^{n-1} |r_{i+1} r_i^{-1}| = \left(\prod_{k=0}^{j-1} \prod_{i \in I_k} |r_{i+1} r_i|\right) \\
&\qquad \left(\prod_{i \in I_j, i<n} |r_{i+1} r_i|\right) \\
&\leq \prod_{k=0}^{j} \left(1 + \frac{1}{4}\sqrt[4]{40}\, \pi (k+j_f)^{-\frac{3}{2}} \right. \\
&\qquad \left. + C_2 (k+j_f)^{4p-2}\right)^N \\
&\leq \prod_{k=0}^{\infty} \left(1 + \frac{1}{4}\sqrt[4]{40}\, \pi (k+j_f)^{-3/2} \right. \\
&\qquad \left. + C_2 (k+j_f)^{4p-2}\right)^N.
\end{aligned}$$

Since $4p - 2 < -1$, this infinite product converges. So, the above inequality immediately yields the assertion of the lemma.

We note that an expression of upper bound for $|u(t) f(t)^{1/4}|$ is given as follows:

Proposition 7. *Define constants T', M, C, N, and C_f by*

$$\begin{aligned}
T' &= T + 2\pi f(T)^{-1}, \\
M &= \max_{T \leq t \leq T'} |u(t)|, \\
N &= 1 + \lfloor \sqrt{30} f'(T)^2 f(T)^{-2(p+1)} [5A\pi]^{-1} \rfloor, \\
C &= 3\sqrt[4]{40}\, \pi/4 + (10 - 20p)(40)^{2-4p} \\
&\qquad f(T)^{8-22p} f'(T)^{16p-6}/(8 - 32p), \\
C_f &= 2\pi \exp(CN) f(T')^2 f(T)^{-3/2}.
\end{aligned}$$

Then, $|u(t) f(t)^{1/4}| \leq M C_f$ for $t \geq T'$.

Remark. The family $\mathscr{F}_{a,p,A}$ is large enough for certain applications. Put $\mathscr{P} = \{P \in \boldsymbol{R}[t] \mid P(t) = \sum_{j=0}^{n} a_j t^j$ with $n > 0$ and $a_n > 0\}$, then we have the following facts for $\mathscr{F}_{a,p,A}$.

(i) For any $P = \sum_{j=0}^{n} a_j t^j \in \mathscr{P}$, $\exp(P(\cdot)) \in \mathscr{F}_{a,p,a_n/2}$ for sufficiently large a.

(ii) Let φ be a polynomial in $\mathscr{P}$ with $\deg \varphi \geq 2$ and f be a smooth function in $\mathscr{F}_{a,p,A}$. Then, $\varphi \circ f \in \mathscr{F}_{a',p,1}$ for sufficiently large a'.

(iii) Assume $f \in \mathscr{F}_{a,p,A}$ and $\varphi \in \mathscr{F}_{b,q,B}$. Then, for every number $r > q$, $\varphi \circ f \in \mathscr{F}_{c,r,1}$ for sufficiently large c.

3. Example. Take the coefficient function f in equation (1) appropriately, then our result gives well-known estimates for all the solutions of Bessel's differential equation.

Here we give an example concerning our work [3]. In that paper, we studied the embeddings of discrete series representations into generalized principal series representations, for a normal real form of a connected, simply connected, complex simple Lie group of type G_2. Such an embedding is determined by observing the structure of the solution space of a certain differential equations [2]. Since discrete series representations are unitary, we are especially interested in their unitary embeddings into (unitarily) induced representations. To determine unitary embeddings, we should examine some conditions, for example, square integrability of functions in that solution space. So, evaluation of decay of solutions of differential equations is of importance in this problem. In the computation of embeddings, a differential equation reduced to the following one has appeared:

(3) $$u''(t) + e^t u(t) = 0.$$

Let ϕ_j $(j = 1, 2)$ be analytic functions defined respectively by

$$\phi_1(t) = \sum_{n=0}^{\infty} (-1)^n (n!)^{-2} t^n,$$

$$\phi_2(t) = \sum_{n=1}^{\infty} 2\left(\sum_{m=1}^{n} \frac{1}{m}\right) \frac{(-1)^n}{(n!)^2} t^n,$$

and put $\varphi_1(t) = \varphi_1(e^t)$ and $\varphi_2(t) = \varphi_2(e^t) - t\phi_1(e^t)$. Then φ_1 and φ_2 are linearly independent solutions of (3). By Remark above, we see that out theorem is applicable for φ_1 and φ_2. In gives that $\varphi_j(t) = O(e^{-t/4})$ for $j = 1, 2$. Using this estimate, we obtain in turn $\phi_1(t) = O(t^{-1/4})$ and $\phi_2(t) = O(t^{-1/4} \log t)$ as $t \to \infty$.

Details of this work will appear elsewhere.

References

[1] L. Brillouin: The B.W.K. Approximation and Hill's equation. II. Quart. Appl. Math., **7**, 363–380 (1949).

[2] H. Yamashita: Embeddings of discrete series into induced representations of semisimple Lie groups. I. Japan. J. Math., **16**, 31–95 (1990).

[3] T. Yoshinaga: Embeddings of discrete series and related differential equations. Master Thesis, Kyoto Univ. (1996).

[4] A. Wintner: A norm criterion for non-oscillatory differential equations. Quart. Appl. Math., **6**, 183–185 (1948).

[5] A. Wintner: A criterion of oscillatory stability. Quart. Appl. Math., **7**, 115–117 (1949).

[6] A. Wintner: On an inequality of Liapounoff. Quart. Appl. Math., **16**, 175–178 (1959).