The Generalized Whittaker Functions for Several Admissible Representations of $Sp(2, \mathbf{R})$

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Introduction. We study generalized Whittaker functions attached to various types of admissible representations of the real symplectic group of rank 2, including the large discrete series. We characterize the A-radial parts of those functions by using the differential operators introduced by Schmid. Integral expressions are given for the A-radial parts and we calculate their Mellin transforms. The results relate to the archimedean factors of Andrianov's L-functions attached to non-holomorphic automorphic representations of the symplectic group.

§1. Generalized Whittaker functions for an admissible representation. (1.1) Let $G = Sp(2, \boldsymbol{R}) = \{g \in SL_4(\boldsymbol{R}) \mid {}^t gJg = J\}$ be the real symplectic group with split rank 2, where $J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}$. Take the Siegel maximal parabolic subgroup P of G with Levi decomposition $P = L \ltimes N$. Its abelian unipotent radical is given by $N = \left\{ n = \begin{pmatrix} 1_2 & T \\ 0 & 1_2 \end{pmatrix} \mid {}^t T = T \in M_2(\boldsymbol{R}) \right\}$. Fix a unitary character η of N. It corresponds to a real 2×2 symmetric matrix $H_\eta = \begin{pmatrix} h_1 & h_3/2 \\ h_3/2 & h_2 \end{pmatrix}$ and we assume this is non-degenerate. Also take a character χ of the identity component $SO(\eta)$ of the stabilizer subgroup of η in L, where L acts on the character group $\hat{N}$ by the action induced from the action of L on N. By the definitions we can make a well defined character $\chi \cdot \eta$ of the group $R = SO(\eta) \ltimes N$ by multiplying the values of both characters. Then we define a C^∞-induced representation $\mathrm{Ind}_R^G(\chi \cdot \eta) = \{f : G \to \boldsymbol{C} \mid C^\infty, f(rg) = \chi \cdot \eta(r) f(g), r \in R, g \in G\}$ with right G-translation. This is called a reduced generalized Gelfand-Graev representation of G, [7] and [8].

Fix K a maximal compact subgroup of G; $K \simeq U(2)$. Write $\mathfrak{g}_0$, $\mathfrak{k}_0$ the Lie algebras of G, K, and $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\boldsymbol{R}} \boldsymbol{C}$. Given an admissible $(\mathfrak{g}, K)$-module (π, H_π) we define the following space.

(1.2) **Definition.** *We call* $W_{\chi.\eta}(\pi) := \mathrm{Hom}_{(\mathfrak{g},K)}(H_\pi, \mathrm{Ind}_R^G(\chi \cdot \eta))$ *the space of algebraic generalized Whittaker functions for* (π, H_π).

(1.3) For a functional $\Phi \in W_{\chi.\eta}(\pi)$, we consider its restriction to a K-type occurring in $\pi|_K$; for an embedding $\iota : V_\tau \to H_\pi$, $(\tau, V_\tau) \in \hat{K}$, we study the function $\iota^*\Phi(v)(g) = \langle v, \phi_{\pi,\tau}(g) \rangle_K$, with the canonical pairing $\langle\ ,\ \rangle_K$ on $V_\tau \times V_{\tau^*}$, τ^* the contragradient to τ. We call the V_{τ^*}-valued function $\phi_{\pi,\tau}$ on G *a generalized Whittaker function with K-type (τ, V_τ) for the admissible representation π.*

§2. Shift operators. (2.1) Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition, then K acts on $\mathfrak{p} = \mathfrak{p}_0 \otimes_{\boldsymbol{R}} \boldsymbol{C}$ by adjoint, $Ad_{\mathfrak{p}}$. For an irreducible representation (τ, V_τ) of K, define $C^\infty(G; \tau) = \{f : G \to \boldsymbol{C} \mid C^\infty, f(gk) = \tau(k^{-1}) f(g), k \in K, g \in G\}$. Take an orthonormal basis $(X_i)_{i \in I}$ of $\mathfrak{p}_0$ with respect to the Killing form. Then Schmid introduced the following right K-equivariant operator ∇.

(2.2) **Definition.** *Define* $\nabla : C^\infty(G; \tau) \to C^\infty(G; \tau \otimes Ad_{\mathfrak{p}})$ *by* $\nabla \phi(g) = \sum_{i \in I} R_{X_i} \phi(g) \otimes X_i$, $\phi \in C^\infty(G; \tau)$, *where* $R_{X_i}(\cdot)$ *denotes the left invariant differential. This is equivariant to the right K-actions. Corresponding to the irreducible decomposition* $Ad_{\mathfrak{p}} \simeq Ad_{\mathfrak{p}_+} \oplus Ad_{\mathfrak{p}_-}$, ∇ *is devided into a sum* $\nabla = \nabla^+ + \nabla^-$.

(2.3) Parameterizing the set $\hat{K}_{\boldsymbol{C}}$ by the dominant integral highest weights $\{(\lambda_1, \lambda_2) \in \boldsymbol{Z}^2 ; \lambda_1 \geq \lambda_2\}$, the irreducible representation $Ad_{\mathfrak{p}_+}$ (resp. $Ad_{\mathfrak{p}_-}$) has the highest weight (2.0) (resp. $(0, -2)$). If a representation τ_λ has the highest weight $\lambda = (\lambda_1, \lambda_2)$, then the tensor representation $\tau_\lambda \otimes Ad_{\mathfrak{p}_+}$ decomposes into irreducible components as $\tau_{\lambda+(2,0)} \oplus \tau_{\lambda+(1,1)} \oplus \tau_{\lambda+(0,2)}$ (some components possibly vanish). $\tau_\lambda \otimes Ad_{\mathfrak{p}_-}$ has a similar decomposition. We consider a composition of $\nabla^\pm$ and the projection onto the each component. We call these composite operators *the shift operators.*

(2.4) *A-radial parts.* The group G has a decomposition $G = RAK$, where the split subgroup A is defined by $A = \{\mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}); a_i > 0\}$. For a generalized Whittaker function $\phi_{\pi,\tau}$ satisfies $\phi_{\pi,\tau}(rgk) = \chi\cdot\eta(r)\tau^*(k^{-1})\phi_{\pi,\tau}(g)$, $r \in R$, $g \in G$, $k \in K$, it is determined by its values on A. Hence we want to characterize $\phi_{\pi,\tau}|_A \in C^\infty(A) \otimes V_{\tau^*}$ by the shift operators.

§3. Generalized principal series representations and large discrete series representations. (3.1) *A generalized principal representation induced from a maximal parabolic subgroup.* Let P_1 be the standard maximal parabolic subgroup of G corresponding to the long simple root. It has the Levi decomposition $P_1 = M_1A_1N_1$; $M_1 \simeq \{\pm 1\} \ltimes SL(2, \boldsymbol{R})$, $A_1 = \{\mathrm{diag}(a, 1, a^{-1}, 1); a > 0\}$, and the unipotent radical N_1 is the 2-step nilpotent group. Take a representation of M_1 consisting of a pair $\sigma = (\varepsilon, D)$, where ε is the trivial or the sign character and D a discrete series representation of $SL(2, \boldsymbol{R})$. Also for $\nu_1 \in \mathfrak{a}_{1\boldsymbol{C}}^*$ we make a character $\exp(\nu_1): A_1 \to \boldsymbol{C}^*$. From these we define a generalized principal series representation $\pi = I(P_1; \sigma, \nu_1)$ as the induced representation $\mathrm{Ind}_{P_1}^G(\sigma \otimes (\nu_1 + \rho_1))$ from P_1 to G ([Kn], Chap. 7, §1). Here ρ_1 is the half sum of positive roots in $\mathfrak{n}_1 = \mathrm{Lie}\, N_1$.

The Frobenius reciprocity gives the K-type decomposition of this representation. Denote $\gamma_{2e_1} = \mathrm{diag}(-1, 1, -1, 1) \in M_1$ and D_k^+ (resp. D_k^-) the holomorphic (resp. anti-holomorphic) discrete series representation of $SL(2, \boldsymbol{R})$ with Blattner parameter $k > 0$ (resp. $k < 0$). We set $\mathrm{sgn}(D_k^+) = +1$, $\mathrm{sgn}(D_k^-) = -1$.

(3.2) **Proposition.** *Let $\pi = I(P_1; \sigma, \nu_1)$, $\sigma = (\varepsilon, D_k^\pm)$, $\nu_1 \in \mathfrak{a}_{1\boldsymbol{C}}^*$, be a generalized principal series representation of G induced from P_1. Then the irreducible representation $\tau_{\lambda_1,\lambda_2}$ of K occurs in the restriction $\pi|_K$ to K with multiplicity*

$$[\pi:\tau_{\lambda_1,\lambda_2}] = \#\left\{m \in \boldsymbol{Z} \,\middle|\, \begin{array}{l} m \equiv k \pmod 2,\ \mathrm{sgn}(D_k^\pm)\cdot(m-k) \geq 0, \\ (-1)^{\lambda_1+\lambda_2-m} = \varepsilon(\gamma_{2e_1}),\ \lambda_2 \leq m \leq \lambda_1 \end{array}\right\}.$$

(3.3) *Large discrete series representations of G.* Take a compact Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ and consider root system $\Sigma(\mathfrak{g}, \mathfrak{h})$. Writing e_1, e_2 bases of $\mathfrak{h}^*$, we have $\Sigma(\mathfrak{g}, \mathfrak{h}) = \{\pm e_1 \pm e_2, \pm 2e_1, \pm 2e_2\}$. If we fix $\{e_1 - e_2\} = \Sigma_c^+$, a compact positive root, then there are $4 = |W(\mathfrak{g}, \mathfrak{h})/W(\mathfrak{k}, \mathfrak{h})|$ choices of positive systems of $\Sigma(\mathfrak{g}, \mathfrak{h})$ compatible with it. One is $\Sigma_{II}^+ = \{e_1 + e_2, 2e_1, e_1 - e_2, -2e_2\}$. Then we can attach a discrete series representation of G, π_Λ, to a dominant weight Λ with respect to Σ_{II}^+. The form Λ is called the Harish-Chandra parameter of π_Λ. We call discrete series representations whose Harish-Chandra parameters are dominant to Σ_{II}^+, the *large* ones (these make a part of the set of all discrete series representations of G).

(3.4) Let π_Λ be a large discrete series representation of G. The K-types of it are known by Blattner formula. For the representation π_Λ we can attach the Blattner parameter $\lambda = (\Lambda_1 + 1)e_1 + \Lambda_2 e_2$. Then the discrete series π_Λ has the minimal K-type $\tau_\lambda = \tau_{(\Lambda_1+1,\Lambda_2)}$ with multiplicity one, and the other possible K-types have the highest weights of the form $\lambda + m_1(e_1 + e_2) + m_2(2e_1) + m_3(-2e_2)$ with $m_1, m_2, m_3 \in \boldsymbol{Z}_{\geq 0}$ occurring with finite multiplicity.

§4. Systems of differential equations for the generalized Whittaker functions attached to admissible Harish-Chandra modules. (4.1) *A generalized Whittaker function for a generalized principal series representation.* After Proposition (3.2), we define the following notion.

(4.2) **Definition.** *Let $\pi = I(P_1; \sigma, \nu_1)$, $\sigma = (\varepsilon, D_k^\pm)$ be a generalized principal series of G. We take its K-finite vectors and make an admissible $(\mathfrak{g}, K)$-module. The following K-type occurring in π is called the corner K-type of π,*
(i) *if $\varepsilon(\gamma_{2e_1}) = (-1)^k$, the K-type $\tau_{(k,k)}$;* (ii) *if $\varepsilon(\gamma_{2e_1}) = -(-1)^k$, the K-type $\tau_{(k,k-1)}$.*
The corner K-type occurs in π with multiplicity one. We call the case (i) (*resp. the case* (ii)) *the even* (*resp. the odd*) *case in the followings.*

(4.3) *A realization of an irreducible representation of K.* We have an isomorphism of $\mathfrak{k}$ to $\mathfrak{u}(2) \otimes \boldsymbol{C}$. A basis of $\mathfrak{u}(2) \otimes \boldsymbol{C}$ given by $Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $H' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\bar{X} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Let $(\tau_\lambda, V_\lambda)$ be a representation of K.

(4.4) **Lemma.** *We have a basis $\{v_j^\lambda \mid 0 \leq j \leq d = \lambda_1 - \lambda_2\}$ of V_λ such that the representation can be given by $\tau_\lambda(Z)v_j^\lambda = (\lambda_1 + \lambda_2)v_j^\lambda$, $\tau_\lambda(H')v_j^\lambda = (2j - d)v_j^\lambda$, $\tau_\lambda(X)v_j^\lambda = (j+1)v_{j+1}^\lambda$, $\tau_\lambda(\bar{X})v_j^\lambda = (d + 1 - j)v_{j-1}^\lambda$.*

(4.5) We can calculate explicitly the Clebsch-Gordan coefficients for the tensor representations

$\tau_\lambda \otimes Ad_{\mathfrak{p}_\pm}$ with the above basis. Calculating the shift operators, we have the following proposition.

(4.6) **Proposition.** *Suppose, for the character η, H_η is positive definite and $h_3 = 0$. Let $\lambda = (k, k)$ or $(k, k-1)$ be the highest weight of the corner K-type of $\pi = I\ (P_1, \sigma, \nu_1)$, and $\phi_{\pi,\tau_\lambda}(a_1, a_2) = \sum_{j=0}^{d} b_j(a_1, a_2) v_j^{\lambda^*}$ (λ^*: the highest weight of the contragradient representation to τ_λ) be an A-radial part of a generalized Whittaker function with the above K-type for π. Then $b_j(a_1, a_2)$ satisfies the following differential equations.*

(i) (*even case*) *Define* $b_0(a_1, a_2) = (\sqrt{h_1}\, a_1)^{k+1} (\sqrt{h_2}\, a_2)^{k+1} e^{-2\pi(h_1a_1^2+h_2a_2^2)} c_0(a_1, a_2)$. *Then the function $c_0(a) = c_0(a_1, a_2)$ has to satisfy*

$$\left(\partial_1\partial_2 - \frac{h_2a_2^2}{\triangle}\partial_1 + \frac{h_1a_1^2}{\triangle}\partial_2 - \mathcal{S}^2\right)c_0(a) = 0, \text{ and}$$

$$\{(\partial_1 + \partial_2)^2 + 2k(\partial_1 + \partial_2) - 8\pi h_1a_1^2\partial_1 - 8\pi h_2a_2^2\partial_2 - 8\pi\,(h_1a_1^2 + h_2a_2^2) + k^2 - \nu_1^2\}\, c_0(a) = 0.$$

(ii) (*odd case*) *Let* $b_j(a_1, a_2) = (\sqrt{h_1}\, a_1)^{k+1+j} (\sqrt{h_2}\, a_2)^{k+2-j}\, e^{-2\pi(h_1a_1^2+h_2a_2^2)} c_j(a_1, a_2),\ j = 0, 1$. *Then $c_j(a) = c_j(a_1, a_2)$, $j = 0, 1$ have to satisfy*

$$\rho\frac{h_2a_2^2}{\triangle}c_0(a) + \left(\partial_2 - \frac{h_2a_2^2}{\triangle}\right) c_1(a) = \left(\partial_1 + \frac{h_1a_1^2}{\triangle}\right) c_0(a) + \rho\frac{h_1a_1^2}{\triangle}c_1(a) = 0, \text{ and}$$

$$(\mathcal{L} + 8\pi(h_1a_1^2 - h_2a_2^2))\, c_0(a) = (\mathcal{L} - 8\pi(h_1a_1^2 - h_2a_2^2))\, c_1(a) = 0.$$

Here $\triangle = h_1a_1^2 - h_2a_2^2$, $\mathcal{S} = \mu\,\dfrac{h_1a_1h_2a_2}{\triangle}$, $\rho = \mu\sqrt{h_1h_2}$, $\mathcal{L} = (\partial_1 + \partial_2)^2 + 2\,(k+1)(\partial_1 + \partial_2) - 8\pi h_1a_1^2\partial_1 - 8\pi h_2a_2^2\partial_2 - 16\pi\,(h_1a_1^2 + h_2a_2^2) + (k+1)^2 - \nu_1^2$ *and μ is the value of χ at* $H_\eta^{-1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{so}(\eta)$.

Proof. About the first equation in the even case we make a composition of 2 shift operators; from K-type $(-k, -k)$ to $(-k+2, -k)$ and from $(-k+2, -k)$ to $(-k+2, -k+2)$. By the Proposition (3.2), we know $\tau_{-k+2,-k+2}$ does not occur in K-types of the principal series. Hence our operator has to annihilate $\phi_{\pi,\tau_{k,k}}$, and it gives the first equation. The second equation is obtained by the action of the Casimir element in $Z(\mathfrak{g})$. The odd case is similarly calculated.

(4.7) *A generalized Whittaker function for a large discrete series representation.* Take a large discrete series representation π_Λ of G with Harish-Chandra parameter Λ, hence with the Blattner parameter $\lambda = (\lambda_1, \lambda_2)$, $\Lambda = (\lambda_1 - 1, \lambda_2)$ (see (3.3), (3.4)).

(4.8) **Proposition.** *Suppose that the matrix H_η is positive definite and $h_3 = 0$ for the character η. Let $\phi_{\pi_\Lambda,\tau_{\lambda_1,\lambda_2}}(a_1, a_2) = \sum_{j=0}^{d} b_j(a_1, a_2) v_j^{-\lambda_2,-\lambda_1}$ be a generalized Whittaker function with the minimal K-type $\tau^*_{\lambda_1,\lambda_2} = \tau_{-\lambda_2,-\lambda_1}$ attached to the large discrete series representation π_Λ. Then it has to satisfy the following differential equations.*

$$\left(\mathcal{L}^1_{j-1} + d\frac{h_2a_2^2}{\triangle}\right)b_{j-1}(a) + 2\mathcal{S}b_j(a) + \left(\mathcal{L}^2_{j+1} - d\frac{h_1a_1^2}{\triangle}\right)b_{j+1}(a) = 0, \text{ for } 1 \le j \le d-1\ ;$$

$$j\mathcal{L}^1_{j-1}b_{j-1}(a) - (d-2j)\mathcal{S}b_j(a) - (d-j)\mathcal{L}^2_{j+1}b_{j+1}(a) = 0, \text{ for } 0 \le j \le d\ ;$$

$$\left(\partial_2 - 4\pi h_2a_2^2 - 2j\frac{h_1a_1^2}{\triangle} + j - 1 + \lambda_2\right) b_{j-1}(a) - 2\mathcal{S}b_j(a) + \left(\partial_1 - 4\pi h_1a_1^2 + 2\,(d-j)\frac{h_2a_2^2}{\triangle} - j - 1 + \lambda_1\right) b_{j+1}(a) = 0, \text{ for } 1 \le j \le d-1,$$

with $\mathcal{L}^1_j = \partial_1 + 4\pi h_1a_1^2 - (d-2j-2)\dfrac{h_2a_2^2}{\triangle} + j - \lambda_1$, $\mathcal{L}^2_j = \partial_2 + 4\pi h_2a_2^2 - (d-2j+2)\dfrac{h_1a_1^2}{\triangle} - j - \lambda_2$.

Proof. These are obtained by calculating the shift operators which annihilate the minimal K-type vectors of π_Λ. The K-types of the large discrete series π_Λ is given in (3.4), the Blattner formula for the discrete series representation.

§5. Solutions and their Mellin transforms.

(5.1) **Theorem.** *Assume the same as in Proposition (4.6). The system of differential equations given in Proposition (4.6) are completely integrable in both cases, even or odd. There exists unique solution for each of them with the following properties. The properties: it is holomorphic at the locus $h_1a_1^2 - h_2a_2^2 = 0$ and rapidly decreases when $a_1, a_2 \to +\infty$. The solution can be expressed by an integral,*

(i) (*even case*) *We have*

$$b_0(a_1, a_2) = const. \times \frac{e^{-\frac{m_0\pi\sqrt{-1}}{2}}}{\Gamma(2m_0+1)} a_1^{k+1} a_2^{k+2} \cdot \int_0^\infty t^{m_0} J_{m_0}(2\pi\sqrt{-1}\,(h_1a_1^2 - h_2a_2^2)\,t) \times F\left(\frac{2-k+2m_0+\nu_1}{2}, \frac{2-k+2m_0-\nu_1}{2}; 2m_0+1; -t\right) e^{-2\pi(h_1a_1^2+h_2a_2^2)(t+1)}\, dt.$$

Here $m_0 = \frac{|\mu|\sqrt{h_1 h_2}}{2} \in \boldsymbol{Z}_{\geq 0}$, $J_\nu(z)$ *is the* ν*-th Bessel function of the first kind and* $F(a, b; c; z)$ *the hypergeometric function. Taking* $h_1 = h_2 = 1$ *and* $m_0 = 0$, *then it has the following Mellin transform;* $\int_0^\infty b_0(\sqrt{a}, \sqrt{a}) a^{s-\frac{3}{2}} \frac{da}{a} =$ *const.* $\times (4\pi)^{-s-k+\frac{1}{2}} \times \Gamma\left(s + \frac{k-1}{2} + \frac{\nu_1}{2}\right) \times \Gamma\left(s + \frac{k-1}{2} - \frac{\nu_1}{2}\right) \times \Gamma\left(s + \frac{1}{2}\right)^{-1}$,
where Re $\left(s + \frac{k-1}{2} \pm \frac{\nu_1}{2}\right) > 0$. *If* $m_0 > 0$, *then the transform is vanishing.*

(ii) (*odd case*) *We have*

$$b_j(a_1, a_2) = \frac{C(\sqrt{-1})^j}{\Gamma(2m_0+2)} (\sqrt{h_1} a_1)^{k+1+j} (\sqrt{h_2}\, a_2)^{k+2-j} \times \int_0^\infty t^{m_0+1} \Big(J_{m_0}\ (2\pi\ \sqrt{-1}\ (h_1 a_1^2 - h_2 a_2^2) t) + (-1)^{j+1}\sqrt{-1}\, J_{m_0+1}(2\pi\sqrt{-1}\,(h_1 a_1^2 - h_2 a_2^2) t)\Big) \times F\left(\frac{2m_0+3+\nu_1-k}{2}, \frac{2m_0+3-\nu_1-k}{2}; 2m_0+2; -t\right) e^{-2\pi(h_1 a_1^2 + h_2 a_2^2)(t+1)} dt,$$

for $j = 0, 1$. *Here* m_0 *is a nonnegative integer determined by* $\rho^2 = \mu^2 h_1 h_2 = -(2m_0+1)^2$, *and* C *is a non-zero constant. Moreover, when* $h_1 = h_2 = 1$ *and* $m_0 = 0$, *we have* $\int_0^\infty b_j(\sqrt{a}, \sqrt{a}) a^{s-\frac{3}{2}} \frac{da}{a} =$ *const.* $\times (4\pi)^{-s-k} \times \Gamma\left(s + \frac{k-1}{2} + \frac{\nu_1}{2}\right) \times \Gamma\left(s + \frac{k-1}{2} - \frac{\nu_1}{2}\right) \times \Gamma(s+1)^{-1}$, $j = 0, 1$ *for* Re $\left(s + \frac{k-1}{2} \pm \frac{\nu_1}{2}\right) > 0$. *They are vanishing if* $m_0 > 0$.

(5.2) **Theorem.** *The system of differential equations given in Proposition* (4.8) *is completely integrable. There is at most one solution which is holomorphic at* $h_1 a_1^2 - h_2 a_2^2 = 0$ *and rapidly decreases as* $a_1, a_2 \to +\infty$. *Also its Mellin transform has the following formula* $\int_0^\infty b_j\ (\sqrt{a}, \sqrt{a}) a^{s-\frac{3}{2}} \frac{da}{a} = C_j \times (4\pi)^{-s-\frac{\lambda_1+\lambda_2+1}{2}} \times \Gamma\left(s + \frac{\lambda_1-1}{2} + \frac{\lambda_2}{2}\right) \times \Gamma\left(s + \frac{\lambda_1-1}{2} - \frac{\lambda_2}{2}\right) \times \Gamma\left(s + \frac{\lambda_1-\lambda_2+1}{2}\right)^{-1}$,
for Re $\left(s + \frac{\lambda_1-1}{2} \pm \frac{\lambda_2}{2}\right) > 0$, $0 \leq j \leq d$. *The ratio between* C_j *can be determined.*

(5.3) The formulas of Mellin transforms given above relate to the archimedean factors of the Andrianov's spinor L-function attached to the automorphic representations having the representations in consideration at their archimedean place.

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