On a Relation Among Toric Minimal Models

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Abstract: Every normal surface singularity has a unique minimal resolution. On the contrary, a minimal terminalization of higher dimensional singularity is not unique. In this note, we prove that there exists a correspondence between minimal terminalizations of a toric canonical singularity and radicals of initial ideals of term order represented by weight vector.

1. Introduction. Every normal surface singularity has a uniquely determined good resolution called minimal resolution, which plays an inportant role in the studying of surface singularities. For singularities of higher dimension, Minimal Model Conjecture tells us that there should exist a *minimal terminalization.*

Definition. A *minimal terminalization* of a germ of singularities X is a projective birational morphism $\pi : Y \to X$ which satisfies the following two conditions:

(1) Y has only $\boldsymbol{Q}$-factorial terminal singularities.

(2) $K_Y \sim \pi^* K_X + \sum a_i E_i, \quad a_i \leq 0$.

We say that π is a *minimal resolution* or a *minimal* $\boldsymbol{Q}$*-factorization* if Y is smooth or has only $\boldsymbol{Q}$-factorial canonical singularities, respectively.

It is known that three dimensional singularities and toric singularities have a minimal terminalization. Minimal terminalizations have many nice properties like as minimal resolutions of surface singularities. In dimension three or higher, however, a minimal terminalization is not unique. In this note, we prove that there exists a correspondence between minimal terminalizations of a toric canonical singularity and radicals of initial ideals of term order represented by weight vector.

Definition. Let $R = \boldsymbol{C}[x_1, \cdots, x_n]$ be a polynomial ring in n variables. Fix $\omega = (\omega_1, \cdots, \omega_n) \in \boldsymbol{R}^n$. For any polynomial $f = \sum c_i x^{\alpha_i}$, we define the *initial form* $\mathrm{in}_\omega(f)$ to be the sum of all terms such that the inner product $\omega \cdot \alpha_i$ is maximal. The *initial ideal* attached to a given ideal I is defined to be the ideal generated by all the initial forms:

$$\mathrm{in}_\omega(I) := \langle \mathrm{in}_\omega(f) : f \in I \rangle.$$

We notice that this ideal is not necessarily to be a monomial ideal.

Our main theorem is the following.

Theorem 1. *Let X be a d-dimensional toric canonical singularity. Then there exists an homogeneous binomial ideal I of $\boldsymbol{C}[x_1, \cdots, x_n]$ which satisfies the following four conditions:*

(1) *The ideal I defines the toric variety defined by the dual fan of the defining of X.*

(2) *There exists a one-to-one correspondence between the minimal $\boldsymbol{Q}$-factorizations and the radicals of initial ideals of weight ω in I such that* $\mathrm{Rad}(\mathrm{in}_\omega(I))$ *is a monomial ideal.*

(3) $\mathrm{Rad}(\mathrm{in}_\omega(I))$ *corresponds to the minimal terminalization of and only if* $\mathrm{Rad}(\mathrm{in}_\omega(I))$ *does not contain* $(1 \leq i \leq n)$.

(4) *If X is a Gorenstein canonical singularity,* $\mathrm{Rad}(\mathrm{in}_\omega(I))$ *corresponds to the minimal resolution if and only if* $\mathrm{Rad}(\mathrm{in}_\omega(I)) = \mathrm{in}_\omega(I)$.

2. Proof of theorem. Let $X = \mathrm{Spec}\,\boldsymbol{C}[\sigma^\vee \cap M]$. Assume that the cone σ is generated by $a_1, \cdots, a_m$. Because X has only canonical singularity, by [5, 1.11], there exists a linear function h such that $h(a_i) = r\ (1 \leq i \leq m)$ and $h(b) \geq r$ for $b \in \sigma \cap N$, where r is a positive integer. Let Δ be a $d-1$-dimensional integral polygon such that

$$\Delta := \{x \in \sigma \mid h(x) = r\}.$$

We define a *regular triangulation* of integral polytope.

Definition. Let Δ be a $d-1$-dimensional

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integral polytope and lattice points $\Delta \cap N = \{a_1, \cdots, a_n\}$. Every sufficiently generic vector $\omega = (\omega_1, \cdots, \omega_n) \in \boldsymbol{R}^n$ defines a triangulation Δ_ω as follows: a subset $\{i_1, \cdots, i_r\}$ is a face of Δ_ω if there exists a vector $c \in \boldsymbol{R}^d$ such that

$a_j \cdot c = \omega_j$ if $j \in \{i_1, \cdots, i_r\}$ and

$a_j \cdot c < \omega_j$ if $j \in \{1, \cdots, n\} \setminus \{i_1, \cdots, i_r\}$.

A triangulation of Δ is called a *regular triangulation* if it coincides with Δ_ω for some $\omega \in \boldsymbol{R}^n$.

Lemma 1. (1) *There exists a one-to-one correspondence between minimal $\boldsymbol{Q}$-factorizations and regular triangulations of Δ.*

(2) *A regular triangulation Δ_ω corresponds to the minimal terminalization if and only if every lattice point in $\Delta \cap N$ forms a one dimensional face of Δ_ω.*

(3) *If X is a Gorenstein singularity, a regular triangulation Δ_ω corresponds to the minimal resolution if and only if every maximal simplex has volume one.*

Proof. First we prove that a minimal $\boldsymbol{Q}$-factorization of X is a toric variety. Let $\pi : Y \to X$ be a minimal $\boldsymbol{Q}$-factorization, D a π-very-ample divisor on Y and Y' a toric minimal terminalization of X. There exists the proper transform D' of D on Y' because Y' is birational to Y and has only terminal singularities. By Matsuki [2, Theorem 5.1, 5.2], after taking a finite sequence of D'-flops we can obtain a toric minimal terminalization $\pi'' : Y'' \to X$ and the π''-nef and π''-big divisor D'' which is the proper transform of D'. A linear system mD'' determines a birational morphism $\nu : Y'' \to Y$ by Base Point Free Theorem [1, Theorem 3-1-1]. Then ν is a toric morphism by [5, Corollary 1.7], and Y is a toric variety.

Thus there exists a one-to-one correspondence between minimal $\boldsymbol{Q}$-factorizations $\pi : Y \to X$ and cone decompositions of σ. From the definition of a minimal $\boldsymbol{Q}$-factorization

(1) $K_Y \sim \pi^* K_X$,

(2) Y has only $\boldsymbol{Q}$-factorial singularities,

the corresponding cone decomposition must come from a triangulation of Δ by [4, 1.11]. We check the projectivity of π. Assume that $\sigma = \cup \sigma_\lambda$ is a cone decomposition attached to a regular triangulation Δ_ω. Let $a_{i_1}, \cdots, a_{i_d}$ be one dimensional generators of a cone σ_λ. From the definition of a regular decomposition, there exists a vector $c_\lambda \in \boldsymbol{R}^d$ such that

$a_i \cdot c_\lambda = \omega_j$ if $j \in \{i_1, \cdots, i_d\}$ and

$a_j \cdot c_\lambda < \omega_j$ if $j \in \{1, \cdots, n\} \setminus \{i_1, \cdots, i_d\}$.

We define a piecewise linear function h on $\sigma = \cup \sigma_\lambda$ as

$$h(x) = x \cdot c_\lambda, \; x \in \sigma_\lambda.$$

Then h is a strictly convex function, and hence π is a projective morphism. On the contrary, if $\pi : Y \to X$ is a projective morphism, there exists a strictly convex piecewise linear function h. Let $\omega = (h(a_1), \cdots, h(a_n))$. A triangulation of Δ corresponding to a cone decomposition of σ is a regular triangulation Δ_ω because h is strictly convex. This complete the proof of (1). The second statement follows from [4,1.11] and (1).

If X is a Gorenstein canonical singularity, there exists a linear function h such that $h(x) = 1$ for every one dimensional generator x of σ and we can take $\Delta = \{x \in \sigma \mid h(x) = 1\}$. Thus we obtain (3). □

Remark. Oda and Park proved in [3] that there exists an one to one correspondence between toric minimal $\boldsymbol{Q}$-factorizations and chambers of seconday fan.

The following lemma describes a relation between regular triangulations and the radicals of initial ideals. This result was essentially obtained by Strumfels ([Theorem 8.3, Corollary 8.4, and Corollary 8.9], [4]). We modifies the statement and the proof of Strumfels for our purpose. The following statement and proof are an arrangement of Strumfels's proofs for our purpose.

Lemma 2. *Let μ be a semigroup homomorphism*

$$\mu : \boldsymbol{N}^n \to \boldsymbol{Z}^d, \quad u = (u_1, \cdots, u_n) \mapsto u_1 a_1 + \cdots + u_n a_n.$$

Let I denote a kernel of a ring homomorphism

$$\mu^\vee : \boldsymbol{C}[x_1, \cdots, x_n] \to \boldsymbol{C}[t_1^\pm, \cdots, t_d^\pm], \quad x_i \mapsto t^{a_i}.$$

Then the ideal I defines the toric variety defined by the dual fan of the defining fan of X and satisfies the following three conditions:

(1) *There exists a one-to-one correspondence between regular triangulations Δ_ω of Δ and the radicals of initial ideals* $\mathrm{Rad}(\mathrm{in}_\omega(I))$ *such that* $\mathrm{Rad}(\mathrm{in}_\omega(I))$ *is a monomial ideal.*

(2) $\mathrm{Rad}(\mathrm{in}_\omega(I))$ *corresponds to the regular triangulation such that every lattice points in $\Delta \cap N$ forms a one dimensional face of Δ_ω if and only if* $\mathrm{Rad}(\mathrm{in}_\omega(I))$ *does not contain $x_i (1 \leq i \leq n)$.*

(3) *If X is a Gorenstein singularity,* $\mathrm{Rad}(\mathrm{in}_\omega(I))$ *corresponds to the regular triangulation such that every maximal simplex has volume one if and only if* $\mathrm{Rad}(\mathrm{in}_\omega(I)) = \mathrm{in}_\omega(I)$.

Proof. Fix a regular triangulation $\Delta_{\omega'}$ of Δ. Let ω be an n-dimensional vector such that $\Delta_\omega = \Delta_{\omega'}$. We prove that

(1) $x^u \in \mathrm{Rad}(\mathrm{in}_\omega(I))$ if and only if $\mathrm{supp}(u)$ is not a face of $\Delta_{\omega'}$.

Let x^u be a monomial and $\mu(u) = b$. Then there exists a cone σ_λ which contains b. We can write

(2) $$b = \sum \eta_i a_i$$
$$\eta_i \geq 0 \text{ if } i \in \{i_1, \cdots, i_d\}$$
$$\eta_i = 0 \text{ if } i \in \{1, \cdots, n\} \setminus \{i_1, \cdots, i_d\},$$

where the a_{i_j} $(1 \leq j \leq d)$ are one dimensional generators of σ_λ. From the definition of a regular triangulation, there exists a d-dimensional vector c_λ such that

$$a_j \cdot c_\lambda = \omega_j \text{ if } j \in \{i_1, \cdots, i_d\} \text{ and}$$
$$a_j \cdot c_\lambda < \omega_j \text{ if } j \in \{1, \cdots, n\} \setminus \{i_1, \cdots, i_d\},$$

where $\omega = (\omega_1, \cdots, \omega_n)$. Then

$$u \cdot \omega = \sum u_i \omega_i \geq \sum u_i (a_i \cdot c_\lambda) = b \cdot c_\lambda$$
$$= \sum \eta_i (a_i \cdot c_\lambda),$$

and the equality holds if and only if $u = \eta$ or, equivalently, $\mathrm{supp}(u)$ is a face of Δ_ω. Thus if $\mathrm{supp}(u)$ is not a face of Δ_ω, there exists an element $x^{mu} - x^{m\eta}$ of I for a suitable multiple of η and $\mathrm{in}_\omega(x^{mu} - x^{m\eta}) = x^{mu}$. Hence $x^u \in \mathrm{Rad}(\mathrm{in}_\omega(I))$, which checks Condition (1). Since the ideal I is generated by binomials $x^u - x^w$ such that $\mu(u) = \mu(w)$, $\mathrm{Rad}(\mathrm{in}_\omega(I))$ is a monomial ideal in view of Condition (1). The proof of (1) is now completed. The second statement immediately follows from Lemma 1 (2) and Condition (1).

By [4, Corollary 8.9], every maximal simplex of Δ_ω has volume one if and only if every generator of $\mathrm{in}_\omega(I)$ is square free, which implies (3). □

Theorem clearly follows from Lemma 1 and 2. Q.E.D.

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