Remark on Upper Bounds for $L(1,\chi)$ *)

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(Communicated by Shokichi IYANAGA, M. J. A., April 12, 1996)

§1. Let k be a real quadratic field of discriminant Δ. Let χ be the non-trivial Dirichlet character of k and $L(s, \chi)$ the L-function attached to χ. In [2], Hua obtained the following upper bound for $L(1, \chi)$:

$$L(1, \chi) \leq \frac{1}{2}\log \Delta + 1.$$

It was shown in [7] that, in the case $\Delta \equiv 1 \pmod 4$

$$L(1, \chi) \leq \frac{1}{2}\log \Delta + \gamma - \frac{1}{2},$$

where $\gamma = 0.57721\ldots$ is Euler's constant. Let $\varepsilon(> 1)$ be the fundamental unit of k and h be the class number of k. From the class number formula, the above upper bounds yield respectively the following inequalities

$$h \log \varepsilon \leq \frac{\sqrt{\Delta}}{4}\log \Delta + \frac{\sqrt{\Delta}}{2},$$

$$h \log \varepsilon \leq \frac{\sqrt{\Delta}}{4}\log \Delta + \sqrt{\Delta}\Big(\frac{\gamma}{2} - \frac{1}{4}\Big).$$

We denote $\frac{\sqrt{\Delta}}{4}\log \Delta + \frac{\sqrt{\Delta}}{2}$ by $H(\Delta)$ and $\frac{\sqrt{\Delta}}{4}\log \Delta + \sqrt{\Delta}\Big(\frac{\gamma}{2} - \frac{1}{4}\Big)$ by $W(\Delta)$, respectively. In the following, we restrict ourselves to the case when Δ is a prime p of the form $4n + 1$. In this case, T. Ono has obtained the following inequality in his paper [6]:

$$\varepsilon^h \leq \frac{2}{\sqrt{p}}(1 + \omega)\Big(1 + \frac{\omega}{2}\Big)\cdots\Big(1 + \frac{\omega}{n}\Big) = \frac{2}{\sqrt{p}}\binom{n + \omega}{n},$$

where $\omega = \frac{1 + \sqrt{p}}{2}$ and $\binom{n + \omega}{n}$ is the generalized binomial coefficient.

Putting $O(p) = \log\left(\frac{2}{\sqrt{p}}\binom{n + \omega}{n}\right)$, we have an upper bound

$$h \log \varepsilon < O(p) = \log 2 - \frac{1}{2}\log p + \sum_{k=1}^{n}\log\Big(1 + \frac{\omega}{k}\Big).$$

In this paper, we shall show $O(p) < H(p)$ for any $p \geq 5$ and $O(p) < W(p)$ for $5 \leq p \leq 661$ and $O(p) > W(p)$ for $p \geq 673$. Since it is obvious that $W(p) < H(p)$ for any $p \geq 5$, we have the following theorem.

Theorem. *With the above notation, we have*

$O(p) < W(p) < H(p)$ *for the cases* $5 \leq p \leq 661$,

$W(p) < O(p) < H(p)$ *for the cases* $p \geq 673$.

§2. Since the gamma function $\Gamma(x)$ is logarithmically convex (see [1]), one can easily show the following lemmas 1, 2 for $0 < s \leq 1$, using the functional equation $\Gamma(x + 1) = x\Gamma(x)$:

Lemma 1. *For any natural number n and any $s > 0$, we have the inequality*

$$\frac{n^s}{\Gamma(1 + s)} \leq \binom{n + s}{n}.$$

Lemma 2. *For any $0 < s \leq n\ (\in \mathbf{N})$, we have*

$$\binom{n + s}{n} \leq \frac{2(2n)^s}{\Gamma(1 + s)}.$$

Combining the fact $n = \frac{p - 1}{4} > \frac{1 + \sqrt{p}}{2} = \omega$ for $p \geq 13$, and the above lemmas, we have

Lemma 3. *For any prime $p = 4n + 1 \geq 13$,*

$$\frac{n^\omega}{\Gamma(1 + \omega)} \leq \binom{n + \omega}{n} \leq \frac{2(2n)^\omega}{\Gamma(1 + \omega)}$$

From Stirling's formula, one knows

$$\frac{e^{\omega - \frac{1}{12\omega}}}{\sqrt{2\pi}\,\omega^{\omega + \frac{1}{2}}} < \frac{1}{\Gamma(1 + \omega)} < \frac{e^\omega}{\sqrt{2\pi}\,\omega^{\omega + \frac{1}{2}}}.$$

From the right hand side inequality in Lemma 3 and the right hand side inequality of this formula, one sees

$$O(p) < \log\left(\frac{4(2n)^\omega e^\omega}{\sqrt{2\pi p}\,\omega^{\omega + \frac{1}{2}}}\right) < H(p) + A(p),$$

where $A(p) = \frac{1}{2}(1 + \log 16 - \log \pi - \log p)$.

Since $A(p)$ is a monotone decreasing function and $A(17) = -0.102\ldots < 0$, we have $O(p) < H(p)$ for $p \geq 17$. In the cases $p = 5$ and 13, a direct

*) Dedicated to Professor Hiroaki Hijikata on his 60th birthday.

computation shows the inequality $O(p) < H(p)$. Hence $O(p) < H(p)$ for any prime $p = 4n + 1 \geq 5$.

On the other hand, from the left hand side inequality of Lemma 3 and the left hand side inequality of Stirling's formula, one sees

$$O(p) > \frac{\sqrt{p}}{4}\log p + \frac{\sqrt{p}}{2}B(p) + C + D(p),$$

where $B(p) = 1 - \log 2 + \log\left(\frac{\sqrt{p}-1}{\sqrt{p}}\right) - \frac{\log p}{\sqrt{p}}$, $C = \frac{1}{2}(1 + \log 2 - \log \pi) = 0.274\ldots$ and $D(p) = \frac{1}{2}\log\frac{\sqrt{p}-1}{\sqrt{p}+1} - \frac{1}{6(\sqrt{p}+1)}$. Since $B(p)$ and $D(p)$ are monotone increasing functions, we have

$$B(1277) = 0.0783\ldots > \gamma - \frac{1}{2} = 0.0772\ldots$$

and $C + D(1277) = 0.241\ldots$, and we obtain $O(p) > W(p)$ for $p \geq 1277$. Finally, a direct computation for $5 \leq p \leq 1249$ shows $O(p) < W(p)$ for any prime $p = 4n + 1 \leq 661$ and $W(p) < O(p)$ for any prime $p = 4n + 1 \geq 673$.

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