Degenerating Problem with Directional Derivative for Quasilinear Elliptic Equations of Second Order

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Abstract: Classical solvability and uniqueness in the Hölder space $C^{2+\alpha}(\bar{\Omega})$ is proved for the oblique derivative problem

$$\sum_{i,j=1}^{n} a^{ij}(x)D_{ij}u + b(x, u, Du) = 0 \quad \text{in } \Omega, \quad \partial u/\partial l \equiv \sum_{i=1}^{n} l^i(x)D_i u = \varphi(x) \quad \text{on } \partial\Omega$$

in the case when the vector field $l(x) = (l^1(x), \ldots, l^n(x))$ is tangential to the boundary $\partial\Omega$ at the points of some non-empty set $S \subset \partial\Omega$, and the nonlinear term $b(x, u, Du)$ grows at most quadratically with respect to the gradient Du.

Key words: Quasilinear elliptic operator; degenerate oblique derivative problem; a priori estimates; Leray-Schauder fixed point theorem.

Introduction. In this paper we study the so-called *oblique derivative problem* firstly posed by H. Poincarè ([15]): *given a domain Ω, find solution of elliptic differential equation in Ω that satisfies boundary condition in terms of directional derivative with respect to vector field l defined on the boundary $\partial\Omega$.* More precisely, we shall be concerned with the problem

$$(1) \quad \begin{cases} \sum_{i,j=1}^{n} a^{ij}(x)D_{ij}u + b(x, u, Du) = 0 & \text{in } \Omega, \\ \partial u/\partial l \equiv \sum_{i=1}^{n} l^i(x)D_i u = \varphi(x) & \text{on } \partial\Omega \end{cases}$$

in the degenerate case, i.e. the vector field $l(x) = (l^1(x), \ldots, l^n(x))$ prescribing the boundary operator becomes tangential to $\partial\Omega$ at the points of some non-empty set S.

The linear tangential problem ($b(x, z, p) = \sum_{i=1}^{n} b^i(x)p_i + c(x)z$) has been very well studied in the last three decades. The pioneering works of Bitsadze [3] and Hörmander [10] indicated the dependence of solvability and uniqueness properties on the way in which the normal component of $l(x)$ changes its sign across S. More precisely, suppose S to be submanifold of $\partial\Omega$ of codimension one, and let $l(x) = \tau(x) + \gamma(x)\nu(x)$, where $\nu(x)$ is the unit outward normal to $\partial\Omega$, and $\tau(x)$ is a tangential field such that $|l(x)| = 1$. There are three possible behaviours of $l(x)$ near the set $S = \{x \in \partial\Omega : \gamma(x) = 0\}$:

a) *$l(x)$ is of neutral type*: $\gamma(x) \geq 0$ or $\gamma(x) \leq 0$;

b) *$l(x)$ is of emergent type*: the sign of $\gamma(x)$ changes from $-$ to $+$ in the positive direction on the τ-integral curves through the points of S;

c) *$l(x)$ is of submergent type*: the sign of $\gamma(x)$ changes from $+$ to $-$ along the τ-integral curves through S.

Hörmander's results were refined by Egorov and Kondrat'ev [5] who proved that the linear problem (1) is of Fredholm type in the case a). Moreover, they showed that either the values of u should be prescribed on S in order to get uniqueness in the case *b*), or to accept jump discontinuity on S in order to have existence in the case *c*). What is the common property of the linear problem (1), independently of the type of $l(x)$, is that a loss of regularity of solution arises in comparison to the regular ($S = \emptyset$) oblique derivative problem.

Later, precise studies were carried out in order to indicate the exact regularity that solution of the linear problem (1) gains on the data both in Sobolev and Hölder spaces [4], [6], [12], [13], [19], [22]-[25], and most recently [7], [8].

The investigations on the quasilinear problem (1) (especially, in the weak nonlinear case $b(x, z, p) = \sum_{i=1}^{n} b^i(x, z)p_i + c(x, z)$) were initiated in the papers [16], [17]. In our previous

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study [18] classical solvability results were proved for (1) both in the cases of neutral and emergent C^∞-vector field $l(x)$ supposing C^∞ structure of the elliptic operator. Moreover, we have assumed that $l(x)$ has a contact of order $k < \infty$ with $\partial\Omega$, and $|b|$, $|b_x| = O(|p|^2)$, $|b_z| = o(|p|^2)$, $|b_p| = o(|p|)$ as $|p| \to \infty$ uniformly on x and z.

The aim of the present article is to improve the results of [18] weakening the growth assumptions on $b(x, z, p)$ with respect to p. Consider at first the case of emergent field $l(x)$. According to the above mentioned result of Egorov and Kondrat'ev, we consider the problem (1) supplied with the additional condition

(2) $\quad u = \psi(x)$ on the set of tangency S.

About the problem (1), (2) we prove solvability and uniqueness in the Hölder space $C^{2+\alpha}(\bar{\Omega})$ assuming $a^{ij} \in C^\alpha(\bar{\Omega})$, $b(x, z, p) \in C^\alpha(\bar{\Omega} \times \boldsymbol{R} \times \boldsymbol{R}^n)$, $l^i \in C^{2+\alpha}(\partial\Omega)$ and $|b(x, y, p)| \leq \mu(|u|)(1 + |p|^2)$ with a non-decreasing function μ (no growth assumptions on the derivatives of b are made !). Further, suitable conditions due to P. Guan and E. Sawyer [8] on the behaviour of $l(x)$ on $\partial\Omega$ are imposed. It is worth noting that our growth condition on $b(x, z, p)$ includes those in [18], as well as the natural structural conditions in the treatment of regular oblique derivative problems for nonlinear elliptic equations (see [11]). The case of neutral field on $\partial\Omega$ will be studied too.

To fix the idea we discuss the problem (1), (2) only. The main tool in proving our results is the Leray-Schauder fixed point theorem, that reduces solvability of (1), (2) to the establishment of a priori $C^{1+\beta}(\bar{\Omega})$-estimate for the solutions of related problems. The bound for $\|u\|_{C^0(\bar{\Omega})}$ is a consequence of the maximum principle. In order to estimate the $C^\beta(\bar{\Omega})$-norm of the gradient Du we use an approach due to F. Tomi [20](see [1] also) that imbeds the problem (1), (2) into a family of similar problems depending on a parameter $\rho \in [0,1]$ and having solutions $u(\rho\,; x)$. Then the norm $\|Du\|_{C^\beta(\bar{\Omega})} = \|D_x u(1\,; x)\|_{C^\beta(\bar{\Omega})}$ can be estimated in terms of $\|D_x u(0\,; x)\|_{C^\beta(\bar{\Omega})}$ after iterations on ρ, assuming the difference $u(\rho_1\,; x) - u(\rho_2\,; x)$ to be under control for small $\rho_1 - \rho_2$. To realize this strategy, we use the refined sub-elliptic estimates in Sobolev and Hölder spaces proved very recently by P. Guan and E. Sawyer [8]. At the end, uniqueness for the solutions of (1), (2) follows from the maximum principle.

1. Statement of the problem and main results. Let $\Omega \subset \boldsymbol{R}^n$, $n \geq 2$, be a bounded domain. On the boundary $\partial\Omega$ a unit vector field $l(x) = (l^1(x), \ldots, l^n(x))$ is defined, which can be decomposed into

$$l(x) = \tau(x) + \gamma(x)\nu(x) \quad x \in \partial\Omega,$$

where $\nu(x)$ is the unit outward normal to $\partial\Omega$ and $\tau(x)$ is the tangential projection of $l(x)$ on $\partial\Omega$. Let

$$S = \{x \in \partial\Omega : \gamma(x) = 0\}$$

be the set of tangency between $l(x)$ and $\partial\Omega$. Throughout the paper we consider the case $S \not\equiv \emptyset$. In order to describe our technique, we shall consider the case of emergent field $l(x)$ only. In other words, we suppose that the sign of the normal component $\gamma(x)$ changes from $-$ to $+$ in the positive direction on the integral curves of the field $\tau(x)$ through the points of S. Moreover, to avoid unessential complications, we assume that S is a closed submanifold of $\partial\Omega$, $\operatorname{codim}_{\partial\Omega} S = 1$, $\partial\Omega = \partial\Omega_+ \cup \partial\Omega_- \cup S$ where $\partial\Omega_\pm = \{x \in \partial\Omega : \gamma(x) \gtrless 0\}$, and let the field $l(x)$ be strictly transversal to S at each point $x \in S$ (indeed, $l \equiv \tau$ there).

We aimed at the investigation of the classical solvability of the degenerate oblique derivative problem:

$$(3)\quad \begin{cases} a^{ij}(x)D_{ij}u + b(x, u, Du) = 0 & \text{in } \Omega, \\ \partial u/\partial l \equiv l^i(x)D_i u = \varphi(x) & \text{on } \partial\Omega, \\ \qquad\qquad\qquad u = \psi(x) & \text{on } S. \end{cases}$$

Hereafter, the standard summation convention is adopted and $Du = (D_1 u, \ldots, D_n u)$ is the gradient of the function $u(x)$, where $D_i \equiv \partial/\partial x_i$. Further, the symbol $C^q(\bar{\Omega})$, $q > 0$ non-integer, stands for the Hölder space equipped with the norm $\|\cdot\|_{C^q(\bar{\Omega})}$ (see [9]). The letter C will denote a constant, independent of u, that may vary from line to line.

In order to state our result, we give a list of assumptions.

Uniform ellipticity: there exists a positive constant λ such that

$$(4)\quad a^{ij}(x)\xi^i\xi^j \geq \lambda|\xi|^2 \quad \forall x \in \bar{\Omega},\ \forall \xi \in \boldsymbol{R}^n, \qquad a^{ij} = a^{ji};$$

Regularity conditions: for some $\alpha \in (0,1)$

(5) $\begin{cases} a^{ij} \in C^{\alpha}(\bar{\Omega}), \\ b(x, z, p) \in C^{\alpha}(\bar{\Omega} \times \boldsymbol{R} \times \boldsymbol{R}^{n}), \\ b(x, z, p) \text{ is continuously differentiable} \\ \text{with respect to } z \text{ and } p, \\ l^{i}(x) \in C^{2+\alpha}(\partial\Omega),\ \partial\Omega \in C^{3+\alpha},\ S \in C^{2+\alpha}; \end{cases}$

Monotonicity condition: there exists a positive constant b_0 such that

(6) $b_z(x, z, p) \leq -b_0 < 0 \quad \forall (x, z, p) \in \bar{\Omega} \times \boldsymbol{R} \times \boldsymbol{R}^{n}$;

Quadratic growth with respect to the gradient: there exists positive and non-decreasing function $\mu(t)$ such that

(7) $$|b(x, z, p)| \leq \mu(|z|)(1 + |p|^2) \quad \forall (x, z, p) \in \bar{\Omega} \times \boldsymbol{R} \times \boldsymbol{R}^{n}.$$

Let $\omega(t, x)$ be the parametrization of the τ-integral curve passing through the point $x \in \partial\Omega$, i.e. $\frac{d}{dt}\omega(t, x) = \tau(\omega(t, x))$, $\omega(0, x) = x$.

The next notions were introduced by P. Guan and E. Sawyer in [8].

Definition 1. The vector field $l(x)$ satisfies condition $\mathscr{A}_p^{\mp}$ on S if for each $y \in S$ there exist positive constants $r > 0$, $R^- < 0 < R^+$ such that $\gamma(\omega(R^-, x)) \neq 0$, $\gamma(\omega(R^+, x)) \neq 0$ for all $x \in S$, $|x - y| < r$ and both of the following conditions hold:

$$\left[\frac{1}{\int_{t_1}^{t_2} \gamma(\omega(t, x))\, dt} \int_{t_1}^{t_2} \gamma(\omega(t, x))^{\frac{p}{p-1}}\, dt\right]^{p-1} \leq C \frac{1}{t_3 - t_2} \int_{t_2}^{t_3} \gamma(\omega(t, x))\, dt$$

for all $x \in S$, $|x - y| < r$ and all $0 < t_1 < t_2 < t_3 < R^+$ with $\int_{t_1}^{t_2} \gamma(\omega(t, x))\, dt = \int_{t_2}^{t_3} \gamma(\omega(t, x))\, dt$, and also

$$\left[\frac{1}{\int_{t_2}^{t_3} |\gamma(\omega(t, x))|\, dt} \int_{t_2}^{t_3} |\gamma(\omega(t, x))|^{\frac{p}{p-1}}\, dt\right]^{p-1} \leq C \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |\gamma(\omega(t, x))|\, dt$$

for all $x \in S$, $|x - y| < r$ and all $R^- < t_1 < t_2 < t_3 < 0$ with

$$\int_{t_1}^{t_2} |\gamma(\omega(t, x))|\, dt = \int_{t_2}^{t_3} |\gamma(\omega(t, x))|\, dt.$$

Definition 2. The vector field $l(x)$ satisfies the condition $\mathscr{T}_\theta$ if

$$t_2 - t_1 \leq C \left(\int_{t_1}^{t_2} |\gamma(w(t, x))|\, dt\right)^{\theta}$$

for all $t_1 < t_2$ and $x \in \partial\Omega$.

Our final assumption concerns the behaviour of $l(x)$ on $\partial\Omega$:

(8) $\begin{cases} \text{The vector field } l(x) \text{ satisfies conditions} \\ \mathscr{A}_q^{\mp} \text{ and } \mathscr{T}_\theta \\ \text{for some } q > n \text{ and } \theta \in [0, 1),\ \theta \neq \alpha. \end{cases}$

We are in a position now to state the main result of the paper.

Theorem 1. *Suppose assumptions* (4) – (8) *to be fulfilled.*

Then the degenerate oblique derivative problem (3) *admits a unique classical $C^{2+\alpha}(\bar{\Omega})$ solution for each $\varphi \in C^{2+\alpha-\theta}(\partial\Omega)$, $\psi \in C^{2+\alpha}(S)$.*

Remark 1. Requirements in (5) on $b(x, z, p)$ to be diffferentiable with respect to z and p may be replaced by the Lipschitz continuity in z and p.

2. Quadratic growth assumption (7) includes for example the natural conditions in studying the regular oblique derivative problems for fully nonlinear elliptic operators (cf. [11]), as well as the structure conditions on $b(x, z, p)$ imposed in [18].

3. Conditions $\mathscr{A}_p^{\mp}$ and $\mathscr{T}_\theta$ correspond to the requirement of "finite type" vector field l in the C^∞ case (cf. [6], [7], [18]). In fact, supposing $\partial\Omega \in C^\infty$, $l \in C^\infty$, we say that the field $l(x)$ is of finite type if there exists an integer k, such that

$$\sum_{i=1}^{k} \left| \frac{\partial^i}{\partial t^i} \gamma(\omega(t, x)) \Big|_{t=0} \right| > 0 \text{ for all } x \in \partial\Omega.$$

Indeed, the number k is exactly the order of contact between the field l and $\partial\Omega$.

Now, if l is of type k, then Lemma C.1 in [21] implies condition $\mathscr{T}_\theta$ with $\theta = \frac{1}{k+1}$.

Moreover, it follows from [7] that $\mathscr{A}_p^{\mp}$ condition is satisfied for all p in the range $(1, \infty)$.

4. Due to the tangency of $l(x)$ to $\partial\Omega$ the condition $\mathscr{T}_\theta$ is valid with $\theta < 1$.

In the case of neutral vector field $l(x)$, the following result holds true.

Theorem 2. *Let assumptions* (4)-(8) *be fulfilled. Suppose further that $l(x)$ is of neutral type and $\boldsymbol{K}(x)\gamma(x) \geq 0$ on $\partial\Omega$, $|\boldsymbol{K}(x)| \geq \boldsymbol{K}_0 = const > 0$ on $\partial\Omega$ with $\boldsymbol{K} \in C^{2+\alpha}(\partial\Omega)$.*

Then the degenerate oblique derivative problem

$$\begin{cases} a^{ij}(x)D_{ij}u + b(x, u, Du) = 0 & \text{in } \Omega, \\ \partial u/\partial l + \boldsymbol{K}(x)u = \varphi(x) & \text{on } \partial\Omega \end{cases}$$

admits a unique classical solution $u \in C^{2+\alpha}(\bar{\Omega})$ for each $\varphi \in C^{2+\alpha-\theta}(\partial\Omega)$.

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