The Embeddings of Discrete Series into Principal Series for an Exceptional Real Simple Lie Group of Type G_2

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Let G_C be a connected, simply connected, complex simple Lie group of type G_2, G its normal real form, and K a maximal compact subgroup of G. In this paper, we give a complete description of the embeddings of discrete series representations of G into principal series. The result is Theorem 2 in §6

1. Structures of G and its Lie algebra. Let G, K be as above, $\mathfrak{g}_0$ (resp. $\mathfrak{k}_0$) the Lie algebra of G (resp. K), $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ a Cartan decomposition of $\mathfrak{g}_0$. Denote by $\mathfrak{g}$ (resp. $\mathfrak{k}$, $\mathfrak{p}$) the complexification of $\mathfrak{g}_0$ (resp. $\mathfrak{k}_0$, $\mathfrak{p}_0$). We take a compact Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{g}_0$ and denote the root system of $\mathfrak{g}$ relative to $\mathfrak{t}(= \mathfrak{t}_0 \otimes \boldsymbol{C})$ by Δ. Let Δ_c (resp. Δ_n) be the set of compact (resp. noncompact) roots and α_1 (resp. α_2) a short (resp. long) simple root in Δ. We may assume that α_1 is compact, that α_2 is noncompact and that $\Delta_c^+ = \{\alpha_1, 3\alpha_1 + 2\alpha_2\}$. We can take root vectors E_{ij} in the root subspace for the root $i\alpha_1 + j\alpha_2 \in \Delta$ in the following way:

$$B(E_{ij}, E_{-i,-j}) = 2/|i\alpha_1 + j\alpha_2|^2, \quad E_{-i,-j} = -\overline{E_{ij}},$$
$$[E_{10}, E_{01}] = E_{11}, \quad [E_{10}, E_{11}] = 2E_{21},$$
$$[E_{10}, E_{21}] = 3E_{31}, \quad [E_{32}, E_{-3,-1}] = E_{01},$$

where $B(\cdot,\cdot)$ is the Killing form of $\mathfrak{g}$ and $\bar{X}$ is the complex conjugate of X relative to the compact real form $\mathfrak{k}_0 \oplus \sqrt{-1}\,\mathfrak{p}_0$ of $\mathfrak{g}$. Set $H_{ij} = [E_{ij}, E_{-i,-j}]$. Equip $\mathfrak{g}$ with the inner product $(\cdot,\cdot)$ defined by $(X, Y) = -B(X, \bar{Y})$. Define a subspace $\mathfrak{a}_0$ of $\mathfrak{g}_0$ as $\mathfrak{a}_0 = \boldsymbol{R}(E_{01} + E_{0,-1}) + \boldsymbol{R}(E_{21} + E_{-2,-1})$, then $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{p}_0$, and equip $\mathfrak{a}_0^*$ with the lexicographic order relative to the ordered basis $(E_{01} + E_{0,-1}, E_{21} + E_{-2,-1})$ of $\mathfrak{a}_0$. Let Ψ be the system of restricted roots of $\mathfrak{g}_0$ with respect to $\mathfrak{a}_0$ and Ψ^+ a positive system of Ψ. Then we have an Iwasawa decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ (resp. $G = KAN$) of $\mathfrak{g}_0$ (resp. G). We see that $\mathfrak{k}_0 \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{k} = \mathfrak{sl}(2, \boldsymbol{C}) \oplus \mathfrak{sl}(2, \boldsymbol{C})$. The root system Δ_c is of type $A_1 \oplus A_1$, and direct computations give that $K \simeq (SU(2) \times SU(2))/D$ with $D = \{1, (-1_2, -1_2)\}$, where 1_2 is the unit matrix of degree 2.

Let M be the centralizer of A in K, then $M = \{1, m_1, m_2, m_1m_2\}$ with

$$m_1 = \left(\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix} \right)^{\ddagger},$$
$$m_2 = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)^{\ddagger},$$

where $g^{\ddagger}$ is the image of $g \in SU(2) \times SU(2)$ under the covering homomorphism of $SU(2) \times SU(2)$ onto K. Define a unitary character $\sigma_{\varepsilon_1,\varepsilon_2}$ of M through $\sigma_{\varepsilon_1,\varepsilon_2}(m_j) = \varepsilon_j$ for $j = 1,2$, then $\hat{M} = \{\sigma_{\varepsilon_1,\varepsilon_2} \mid \varepsilon_j = \pm 1 (j = 1,2)\}$. For each $\mu \in \mathfrak{a}^* = \mathfrak{a}_0^* \otimes \boldsymbol{C}$ gives an one-dimensional representation e^{μ} of the vector group $A = \exp \mathfrak{a}_0$. Put $P = MAN$ and we consider the principal series $\mathrm{Ind}_P^G(\sigma_{\varepsilon_1,\varepsilon_2} \otimes e^{\mu} \otimes 1_N)$, of G induced from the minimal parabolic subgroup P.

2. Irreducible K-modules. Let X, Y, H be elements in $\mathfrak{sl}(2, \boldsymbol{C})$ with $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$. The $(d+1)$-dimensional irreducible $\mathfrak{sl}(2, \boldsymbol{C})$-module is denoted by V_d. Take a basis $\{e_p^{(d)} \mid p = -d, -d+2, \ldots, d\}$ of V_d satisfying the relation

$$\begin{cases} H \cdot e_p^{(d)} = p e_p^{(d)} \\ X \cdot e_p^{(d)} = x_p^{(d)} e_{p+2}^{(d)} \quad (p = -d, -d+2, \ldots, d) \cdot \\ Y \cdot e_p^{(d)} = x_{p-2}^{(d)} e_{p-2}^{(d)} \end{cases}$$

Here, $x_p^{(d)} = \frac{1}{2}\sqrt{(d-p)(d+p+2)}$. We regard $e_p^{(d)}$ as 0 if $p \notin \{-d, -d+2, \ldots, d\}$. For a Δ_c^+-dominant, integral linear form λ on $\mathfrak{t}$, put nonnegative integers r, s as $r = \lambda(H_{10})$, $s = \lambda(H_{32})$. The finite-dimensional irreducible representation of K with highest weight λ is denoted by $(\tau_\lambda, V_\lambda)$. Then $V_\lambda \simeq V_r \hat{\otimes} V_s$. Here $\hat{\otimes}$ means an exterior tensor product. So we identify these two modules and take a basis $\{e_{pq}^{(rs)}\}$ of $V_r \hat{\otimes} V_s$. Here $e_{pq}^{(rs)} = e_p^{(r)} \otimes e_q^{(s)}$. Note that $\mathfrak{p} \simeq V_3 \hat{\otimes} V_1$ as K-modules.

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3. Gradient type differential operators. The K-module $V_\lambda \otimes \mathfrak{p}$ decomposes as $V_\lambda \otimes \mathfrak{p} \simeq \oplus_{\beta \in \Delta_n} m(\beta) \cdot V_{\lambda+\beta}$, with multiplicity $m(\beta) = 0,1$ for $\beta \in \Delta_n$. Take a positive system Δ^+ of Δ containing Δ_c^+, and put $V^- = \oplus_{\beta \in \Delta_n^+} m(-\beta) \cdot V_{\lambda-\beta}$, where $\Delta_n^+ = \Delta^+ \cap \Delta_n$ is the set of positive noncompact roots. Let P_λ be the orthogonal projection of $V_\lambda \otimes \mathfrak{p}$ onto V^-. For a representation (τ, V) of K, define two function spaces $C_\tau^\infty(G)$ and $C_\tau^\infty(G\,;1_N)$ as

$$C_\tau^\infty(G) = \{f : G \xrightarrow{c^\infty} V \mid f(kg) = \tau(k) f(g) \\ (\forall\,(k, g) \in K \times G)\},$$

$$C_\tau^\infty(G\,;1_N) = \{f : G \xrightarrow{c^\infty} V \mid f(kgn) = \tau(k) f(g) \\ (\forall\,(k, g, n) \in K \times G \times N)\}.$$

We define a gradient-type differential operator $\mathscr{D}_\lambda$ on $C_{\tau_\lambda}^\infty(G)$ by

$$(\nabla f)(g) = \sum_j L_{X_j} f(g) \otimes \bar{X}_j,$$

$$(\mathscr{D}_\lambda f)(g) = P_\lambda(\nabla f(g)),$$

where L_X is the differentiation with respect to the right invariant vector field on G defined by an element X in $\mathfrak{g}$ and $\{X_j\}$ is an orthonormal basis of $\mathfrak{p}$ relative to the inner product $(\cdot\,,\,\cdot)$. Put $\mathscr{D}_{\lambda,1_N} = \mathscr{D}_\lambda|_{C_{\tau_\lambda}^\infty(G\,;1_N)}$.

4. Parametrization of discrete series of G. Let Ξ_c be the totality of Δ_c^+-dominant,regular, integral linear forms Λ on $\mathfrak{t}$. For each $\Lambda \in \Xi_c$, Δ^+ denotes the positive system of Δ for which Λ is Δ^+-dominant. By Harish-Chandra [1, Theorem 16], discrete series representations of G is parametrized by Ξ_c and we denote the discrete series of G with Harish-Chandra parameter Λ by π_Λ. Let $\Delta_J^+ (J = I, II, III)$ be positive systems of Δ with simple roots listed below:

J	I	II	III
simple roots	α_1, α_2	$\alpha_1 + \alpha_2, -\alpha_2$	$-\alpha_1 - \alpha_2, 3\alpha_1 + 2\alpha_2$

For a discrete series π_Λ of G, the corresponding positive system $\Delta^+ = \{\alpha \in \Delta \mid (\alpha, \Lambda) > 0\} \subset \Delta$ is one of the above Δ_J^+'s. Define three subsets Ξ_J $(J = I, II, III)$ of Ξ_c by $\Xi_J = \{\Lambda \in \Xi_c \mid \Delta^+ = \Delta_J^+\}$. Put $\rho_c = \frac{1}{2} \sum_{\alpha \in \Delta_c^+} \alpha$, $\rho_n = \frac{1}{2} \sum_{\alpha \in \Delta_n^+} \alpha$ and $\lambda = \Lambda - \rho_c + \rho_n$. The discrete series π_Λ has the lowest K-type τ_λ and λ is called the *Blattner parameter* of π_Λ.

5. Method for the determination of embeddings. Take a discrete series π_Λ of G and set Δ^+ as above. The Blattner parameter λ of π_Λ is said to be *far from the walls* if $\lambda - \sum_{\beta \in Q} \beta$ is Δ_c^+-dominant for any subset Q of Δ_n^+. For an irreducible representation $\xi = \sigma \otimes e^\mu$ with $\sigma \in \hat{M}$ and $\mu \in \mathfrak{a}^*$, put $\tilde{\xi} = \sigma \otimes e^{\mu + \rho_P}$. Here $\rho_P \in \mathfrak{a}_0^*$ is defined by $\rho_P(H) = \frac{1}{2} \operatorname{tr} \operatorname{ad}(H)|_{\mathfrak{n}_0}$ for $H \in \mathfrak{a}_0$. It is easily seen that MA acts on $\operatorname{Ker} \mathscr{D}_{\lambda,1_N}$ by right translation. The determination of the embeddings of discrete series into principal series as $(\mathfrak{g}, K)$-modules is based on the following theorem proved for gengeral semisimple Lie groups with finite center.

Theorem 1 (cf. [3, Theorem 3.5]). *If the Blattner parameter λ of π_Λ is far from the walls, then*

$$\operatorname{Hom}_{(\mathfrak{g},K)}(\pi_\Lambda^*, \operatorname{Ind}_P^G(\xi \otimes 1_N)) \simeq \operatorname{Hom}_{(\mathfrak{a},M)}(\tilde{\xi}^*, \operatorname{Ker} \mathscr{D}_{\lambda,1_N}),$$

as linear spaces. Here π_Λ^* denotes the discrete series of G contragredient to π_Λ.

6. Complete descrition of embeddings. Define an automorphism u of $\mathfrak{g}$ by

$$u = \left(\exp \frac{\pi}{4} \operatorname{ad}(E_{01} - E_{0,-1})\right) \cdot \left(\exp \frac{\pi}{4} \operatorname{ad}(E_{21} - E_{-2,-1})\right).$$

Note that u maps $\mathfrak{t}$ onto $\mathfrak{a}$. For $\Lambda \in \Xi_I$, let $\Lambda_J (J = I, II, III)$ be the unique element in $\Delta_J^+ \cap W \cdot \Lambda$, where W is the Weyl group of Δ. Further let $\tilde{\Lambda}$ be the Ψ^+-dominant element in $\mathfrak{a}^*$ conjugate to $\Lambda \circ u^{-1}$ under the action of the Weyl group $W(\Psi)$ of Ψ. Define discrete series representations $\pi_J (J = I, II\ III)$ by $\pi_J = \pi_{\Lambda_J}$. Then these three π_J's are the mutually inequivalent discrete series with the same infinitesimal character Λ. Put λ_j, $j = 1,2$, as $\lambda_1 \circ u = -(2\alpha_1 + \alpha_2)$ and $\lambda_2 \circ u = 3\alpha_1 + \alpha_2$, then these λ_j's are simple roots of Ψ^+. The reflection relative to λ_j is denoted by s_j. The following theorem describes the embeddings of discrete series of G into its principal series.

Theorem 2. *For $\Lambda \in \Xi_I$, $J = I, II, III$, $\sigma_{\varepsilon_1,\varepsilon_2} \in \hat{M}$, $\mu \in \mathfrak{a}^*$,*

$$\dim \operatorname{Hom}_{(\mathfrak{g},K)}(\pi_J, \operatorname{Ind}_P^G(\sigma_{\varepsilon_1,\varepsilon_2} \otimes e^\mu \otimes 1_N)) \leq 1,$$

and the equality holds if and only if
$\mu = s \cdot \tilde{\Lambda}$ and $(\varepsilon_1, \varepsilon_2) \in S_\Lambda(J, s)$ with an $s \in W(J)$, where $W(J)$ and $S_\Lambda(J, s)$ are subsets of $W(\Psi)$ and $\{\pm 1\} \times \{\pm 1\}$ defined respectively as follows:

$$W(I) = \{s_1, s_2 s_1\},$$
$$W(II) = \{1, s_1, s_2, s_1 s_2, s_2 s_1\},$$
$$W(III) = \{s_2, s_1 s_2\},$$

and

$$S_\Lambda(II, 1) = \{((-1)^{\frac{1}{2}(r'+s')}, (-1)^{\frac{1}{2}(r'-s'+2)}), ((-1)^{\frac{1}{2}(r'+s'+2)}, \pm 1)\},$$

$$S_\Lambda(J, s_1) = \begin{cases} \{((-1)^{r'+1}, (-1)^{\frac{1}{2}(r'+s')})\} & \text{for } J = I \\ \{((-1)^{\frac{1}{2}(r'+s')}, (-1)^{r'+1}), ((-1)^{\frac{1}{2}(r'+s'+2)}, \pm 1)\} & \text{for } J = II, \end{cases}$$

$$S_\Lambda(J, s_2) = \begin{cases} \{((-1)^{r'}, (-1)^{\frac{1}{2}(r'-s'+2)}), ((-1)^{r'+1}, \pm 1)\} & \text{for } J = II \\ \{((-1)^{\frac{1}{2}(r'-s')}, (-1)^{r'+1}), ((-1)^{\frac{1}{2}(r'-s'+2)}, (-1)^{r'})\} & \text{for } J = III, \end{cases}$$

$$S_\Lambda(J, s_1s_2) = \{(\pm 1, (-1)^{\frac{1}{2}(r'+s'+2)})\} \quad \text{for } J = II, III,$$

$$S_\Lambda(J, s_2s_1) = \{((-1)^{\frac{1}{2}(r'-s'+2)}, \pm 1)\} \quad \text{for } J = I, II.$$

Here $r' = \Lambda(H_{10})$ *and* $s' = \Lambda(H_{32})$.

7. Sketch of the proof of Theorem 2. Here we illustrate the outline of the proof in case $J = I$, where $r = \lambda(H_{10})$ and $s = \lambda(H_{32})$ are non-negative integers so that $s - r$ is even and is not less than 4. Let f be a function in $C^\infty_{\tau_\lambda}(G; 1_N)$, then f can be expressed uniquely in the form

$$f(g) = \sum_{p,q} c_{pq}(g) e^{(rs)}_{pq},$$

with smooth functions c_{pq} on G. Rewriting the condition $\mathscr{D}_{\lambda,1_N} f = 0$ in terms of c_{pq}'s, we obtain the following system of differential equations:

(7.1) $(2L_1 - 2s + p + q - 2)c_{p,q+2} = 0,$

(7.2) $\sqrt{s-q}\,(2L_2 + p + 3q)c_{pq} + 2\sqrt{(r+2-p)(r+p)(s+2+q)}\; c_{p-2,q+2} = 0,$

(7.3) $-\sqrt{s+2+q}\,(p + 3q + 6 - 2L_2)c_{p,q+2} + 2\sqrt{(r-p)(r+2+p)(s-q)}\; c_{p+2,q} = 0,$

(7.4) $(2s + p + q + 4 - 2L_1)c_{pq} = 0,$

for $p = -r, -r+2, \dots, r$ and $q = -s, -s+2, \dots, s-2$. Here $L_1 = L_{E_{01}+E_{0,-1}}$ and $L_2 = L_{E_{21}+E_{-2,-1}}$.

Since c_{pq}'s are determined by their values on A, we consider the equations for c_{pq}'s such as (7.1)-(7.4) only on A, though c_{pq}'s are functions on G. By (7.1) and (7.4), we have $(p+q)c_{pq} = 0$ if $q \neq \pm s$. So $c_{pq} = 0$ if $q \neq \pm s$ and $p + q \neq 0$. For c_{pq}'s with $q = \pm s$, (7.2) and (7.3) and the previous fact show that $c_{ps} = 0$ if $p \neq r$ and that $c_{p,-s} = 0$ if $p \neq -r$.

To determine the form of the function c_{pq}, define smooth functions $\tilde{c}_{pq}$ on $\boldsymbol{R}^2$ by

$$\tilde{c}_{pq}(x_1, x_2) = c_{pq}(\exp(x_1(E_{01} + E_{0,-1}) + x_2(E_{21} + E_{-2,-1}))),$$

for real numbers x_1, x_2. Then the equations (7.1)-(7.4) give a system of partial differential equations for $\tilde{c}_{pq}$'s. For instance, we can find the following equations for $\tilde{c}_{p,-p}$'s.

(7.5) $-\left(\dfrac{\partial}{\partial x_1} + s + 2\right)\tilde{c}_{p,-p} = 0,$

(7.6) $-\sqrt{s+p}\left(\dfrac{\partial}{\partial x_2} + p\right)\tilde{c}_{p,-p} + \sqrt{(r+2-p)(r+p)(s+2-p)}\; \tilde{c}_{p-2,-(p-2)} = 0,$

(7.7) $\sqrt{s-p}\left(-\dfrac{\partial}{\partial x_2} + p\right)\tilde{c}_{p,-p} + \sqrt{(r-p)(r+2+p)(s+2+p)}\; \tilde{c}_{p+2,-(p+2)} = 0.$

These three equations tell that $\tilde{c}_{p,-p}(x_1, x_2) = a_p \exp(-(s+2)x_1 + rx_2)$ for a scalar constant a_p. In a similar way, c_{rs} and $c_{-r,-s}$ are determined up to scalar multiples. By using (7.6) and (7.7) again, we have the following inductive relation for constants a_p's,

$$a_{p-2} = \sqrt{\frac{(s+p)(r+p)}{(s+2-p)(r+2-p)}}\, a_p,$$

for $p = -r+2, \dots, r$. This implies that $a_p = \alpha_p a_r$ with

$$\alpha_p = \sqrt{\frac{2r(2r-2)\cdots(r+p+2)\cdot(s+r)(s+r-2)\cdots(s+p+2)}{(r-p)(r-p-2)\cdots 2\cdot(s-p)(s-p-2)\cdots(s-r+2)}}.$$

Define linear forms μ_1, μ_2 on $\mathfrak{a}$ through

$$\mu_1(E_{01} + E_{0,-1}) = -(s+2),$$
$$\mu_1(E_{21} + E_{-2,-1}) = r,$$
$$\mu_2(E_{01} + E_{0,-1}) = -(s-r+4)/2,$$
$$\mu_2(E_{21} + E_{-2,-1}) = -(r+3s)/2.$$

Argue as above for c_{rs} and $c_{-r,-s}$, we see that $\operatorname{Ker} \mathscr{D}_{\lambda,1_N}$ is contained in the linear span of the following three functions $f_*(* = 0, +, -)$.

$$f_0(a) = \sum_p \alpha_p a^{\mu_1} e^{(rs)}_{p,-p},$$
$$f_+(a) = a^{\mu_2}(e^{(rs)}_{rs} + e^{(rs)}_{-r,-s}),$$
$$f_-(a) = a^{\mu_2}(e^{(rs)}_{rs} - e^{(rs)}_{-r,-s}),$$

for $a \in A$, where $a^\mu = \exp(\mu(\log a))$, and extend these f_*'s to G through $f_*(kan) = \tau_\lambda(k) f_*(a)$ for $k \in K$, $a \in A$, $n \in N$. It is easily seen that these f_*'s actually form a basis of $\operatorname{Ker} \mathscr{D}_{\lambda,1_N}$.

To see the MA-module structure of $\operatorname{Ker} \mathscr{D}_{\lambda,1_N}$, we decompose it into irreducibles by seeking its suitable basis. In case $J = I$, the subspace $\boldsymbol{C}f_*$ for each of the above three f_*'s is an MA-invariant subspace of $\operatorname{Ker} \mathscr{D}_{\lambda,1_N}$ and as MA-modules

$$\boldsymbol{C}f_0 \simeq (\sigma_{(-1)^r,(-1)^{(r+s)/2}}) \otimes e^{\mu_1},$$
$$\boldsymbol{C}f_+ \simeq (\sigma_{(-1)^{(s-r)/2},1}) \otimes e^{\mu_2},$$
$$\boldsymbol{C}f_- \simeq (\sigma_{(-1)^{(s-r)/2},-1}) \otimes e^{\mu_2}.$$

Applying Theorem 1 to this result for the MA-module structure of $\mathrm{Ker}\,\mathscr{D}_{\lambda,1_N}$ and rewriting the parameters, we can specialize parameters ε_1, ε_2 and μ satisfying

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_\lambda^*,\ \mathrm{Ind}_P^G(\sigma_{\varepsilon_1\varepsilon_2}\otimes e^\mu\otimes 1_N))\neq(0),$$

for discrete series π_Λ whose Blattner parameter is far from the walls. Keeping in mind the fact that each discrete series representation of G is self-contragredient and calculating $s\cdot\tilde{\Lambda}$ for each s in the Weyl group of Ψ, we can varify the assertion in the theorem if the Blattner parameter of π_Λ is far from the walls.

To get rid of the restriction that λ is far from the walls, Zuckerman's translation functors can be used. See [5, Corollary 5.5] and [2, Theorem B.1].

For the case $J=II$ or $J=III$, by similar but more complicated computations, we can derive the statement in the theorem.

The details of this paper will appear elsewhere [4].

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