On $L(1,\chi)$ and Class Number Formula for the Real Quadratic Fields

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1. Introduction. Let k be a positive integer greater than 1, and let $\chi(n)$ be a real primitive character modulo k. The series

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}$$

can be divided into groups of k consecutive terms. Let v be any nonnegative integer, j an integer, $0 \leq j \leq k-1$, and let

$$T(v, j, \chi) = \sum_{n=j+1}^{j+k} \frac{\chi(vk+n)}{vk+n} = \sum_{n=j+1}^{j+k} \frac{\chi(n)}{vk+n}.$$

Then $L(1, \chi) = \sum_{n=1}^{j} \frac{\chi(n)}{n} + \sum_{v=0}^{\infty} T(v, j, \chi)$.

We remind the reader that a real primitive character $(\bmod k)$ exists only when either k or $-k$ is a fundamental discriminant, and that the character is then given by

$$\chi(n) = \left(\frac{d}{n}\right),$$

where d is k or $-k$, and the symbol is that of Kronecker (see, for example, Ayoub [2] for the definition of a Kronecker character).

In [5], Davenport proved the following theorem:

Theorem (H. Davenport). *If* $\chi(-1) = 1$, *then* $T(v, 0, \chi) > 0$ *for all* v *and* k. *If* $\chi(-1) = -1$, *then* $T(0,0, \chi) > 0$ *for all* k, *and* $T(v, 0, \chi) > 0$ *if* $v > v(k)$: *but for any integer* $r \geq 1$ *there exist values of* k *for which*

$$T(1,0, \chi) < 0,\ T(2,0, \chi) < 0, \ldots,\ T(r, 0, \chi) < 0.$$

In [9], Leu and Li derived the following result about $T\left(v, \left[\frac{k}{2}\right], \chi\right)$.

Theorem A. *If* $\chi(-1) = 1$, *then* $T\left(v, \left[\frac{k}{2}\right], \chi\right) < 0$ *for all* v *and* k, *where* $[x]$ *denotes the greatest integer* $\leq x$.

Combining the results of Davenport [5] and Theorem A of Leu and Li, we have the following interesting inequalities.

Theorem B. *If* $\chi(-1) = 1$, *then*

$$(1.1) \qquad \sum_{n=1}^{k} \frac{\chi(n)}{n} < L(1, \chi) < \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}.$$

In section 2, we will prove that $T(v, j, \chi) \neq 0$ for prime integer $k > 2$, nonnegative integer v and $j = 0,1,2, \ldots, k-1$. In section 3, we will derive the inequalities for $L(1, \chi)$ on even real primitive character modulo k:

$$(1.2) \qquad \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n} - \frac{\ln k}{\sqrt{k}} < L(1, \chi) < \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}.$$

On the other hand, using Siegel-Tatuzawa's lower bound for $L(1, \chi)$ (See [15] and [13]) and Louboutin's upper bound for $L(1, \chi)$ (See [10], [3], [14] and [12]), one also has inequalities for $L(1, \chi)$ on even real primitive character modulo k (with one possible exception coming from applying Siegel-Tatuzawa's theorem [15]):

$$(1.3) \qquad \frac{0.655}{4} \frac{1}{k^{1/4}} < L(1, \chi) \leq \frac{1}{2} \ln k + \frac{2 + \gamma - \ln(4\pi)}{2},$$

where $k \geq e^{11.2}$ and γ denotes Euler's constant. From the facts $\lim_{k\to\infty} \frac{\ln k}{\sqrt{k}} = 0$ and $\lim_{k\to\infty} \left(\frac{1}{2} \ln k - \frac{0.655}{4} k^{-\frac{1}{4}}\right) = \infty$, it is clear that the inequalities (1.2) provides much better estimate for $L(1, \chi)$ than the inequalities (1.3) does. In these days, computing facilities are highly developed, the inequalities (1.2) may be used to investigate varied problems related to $L(1, \chi)$. In section 4, we will derive a class number formula for the real quadratic fields:

- If positive integer k is not of the form $m^2 + 4$ $(m \in \mathbf{N})$, then

$$h = \left[\frac{\sqrt{k}}{2 \ln \varepsilon} \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}\right],$$

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where h and $\varepsilon > 1$ denote the class number and the fundamental unit of the real quadratic field $\boldsymbol{Q}(\sqrt{k})$, respectively.

- If $k = m^2 + 4 (m \in \boldsymbol{N}$ and $m \geq 3)$, then

$$h = \left[\frac{\sqrt{k}}{2\ln \varepsilon} \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}\right] - i,$$

where $i = 0$ or $i = 1$ depends on whether $\left[\frac{\sqrt{k}}{2\ln \varepsilon} \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}\right] + E(k)$ is an even integer or an odd integer, where $E(k)$ is 1 or 0 depending on whether k is a prime or not a prime.

2. $T(v, j, \chi) \neq 0$. In order to prove the main result of this section, we need the following propositions and lemmas.

Proposition 1. *Let a_i and b_i be any pair of nonzero integers without any common divisors $(i = 1,2,\ldots, n)$. If there exists a prime number p and a positive integer α such that $p^\alpha \mid b_l$ for some integer l, $1 \leq l \leq n$, and $p^\alpha \nmid b_j$ for any integer j, $1 \leq j \leq n$ and $j \neq l$, then*

$$\sum_{i=1}^{n} \frac{a_i}{b_i} \neq 0.$$

Proof. For integers $i = 1,2,\ldots, n$, we express, by hypothesis, $b_i = p^{\beta_i} m_i$ with m_i an integer without prime factor p and β_i an integer satisfying $\alpha > \beta_i \geq 0$ for $i \neq l$ and $\beta_i \geq \alpha$ for $i = l$. Write $\Pi_{i=1}^{n} b_i = p^t M$, where $t = \beta_1 + \cdots + \beta_n$ and observe that $M = \Pi_{i=1}^{n} m_i$ is an integer without prime factor p. We have

$$\sum_{i=1}^{n} \frac{a_i}{b_i} = \frac{\sum_{i=1}^{n} a_i p^{t-\beta_i} \frac{M}{m_i}}{p^t M} =: \frac{N}{p^t M}.$$

Write the numerator N as a sum of two parts $\sum_{i\neq l} a_i p^{t-\beta_i} \frac{M}{m_i} + a_l \frac{M}{m_l} p^{t-\beta_l}$. Then

$$\begin{aligned} \frac{N}{p^{t-\beta_l}} &= \sum_{i\neq l} a_i p^{\beta_l-\beta_i} \frac{M}{m_i} + a_l \frac{M}{m_l} \\ &\equiv a_l \frac{M}{m_l} \ (mod\, p) \\ &\not\equiv 0 \ (mod\, p) \end{aligned}$$

since a_l and b_l have no common divisors and $\beta_l > 0$, hence $a_l \frac{M}{m_l}$ is an integer without prime factor p. This implies that $N \neq 0$, and therefore $\sum_{i=1}^{n} \frac{a_i}{b_i} \neq 0$. □

Lemma 1. *If $\binom{n}{m}$ is divisible by a power of a prime p^α, then $p^\alpha \leq n$. (The expression $\binom{n}{m} = \frac{n!}{(n-m)!m!}$.)*

Proof. See, e.g., P. Erdös [6, pp. 283], for a proof. □

The following lemma is a result of Hanson [7]:

Lemma 2. *The product of m consecutive integers $n(n+1)\ldots(n+m-1)$ greater than m contains a prime divisor greater than $\frac{3}{2}m$ with the exceptions* $3\cdot 4$, $8\cdot 9$ *and* $6\cdot 7\cdot 8\cdot 9\cdot 10$.

By a simple calculation, we see that $s > \frac{3}{4} m$ for integer $m \geq 4200$, where s is the number of positive integers which are smaller than m and divisible by either 2 or 3 or 5 or 7. In combination with table of prime numbers < 5000, we have the following lemma.

Lemma 3. *For integer $m \geq 50$, $\pi(m) \leq \frac{31}{100} m$, where $\pi(m)$ denotes the number of primes less than or equal to m.*

Applying a method of Hanson [7] and Lemma 3, we have

Proposition 2. *For integer $m \geq 50$ and integer $n \geq m^{\frac{100}{38}}$, $\binom{n}{m}$ has prime factor greater than $2m$.*

Proof. For $m \geq 50$, by Lemma 3, $\pi(m) \leq \frac{31}{100} m$. Suppose that $\binom{n}{m}$ has no prime factor greater than $2m$. Lemma 1 implies

$$\binom{n}{m} \leq n^{\pi(2m)} \leq n^{\frac{31}{100}2m} = n^{\frac{62}{100}m}.$$

However since

$$\binom{n}{m} = \frac{n}{m}\cdot\frac{n-1}{m-1}\cdots\frac{n-m+1}{1} > \left(\frac{n}{m}\right)^m,$$

we must have

$$\left(\frac{n}{m}\right)^m < n^{\frac{62}{100}m}$$

which is false if $m \leq n^{\frac{38}{100}}$, and the proposition follows. □

For any positive odd integer $m > 1$, there exists a unique positive integer α such that $2^\alpha < m < 2^{\alpha+1}$.

Lemma 4. *Let χ be a real primitive character modulo a positive odd prime integer k and α the integer such that $2^\alpha < k < 2^{\alpha+1}$. For any nonnega-*

tive integer v and integer j, $0 \le j \le k-1$, if $(v+1)k$ is not divisible by 2^α, then

$$T(v, j, \chi) \ne 0.$$

Proof. Since $2^\alpha < k < 2^{\alpha+1}$, there is at least one integer, among k consecutive integers $vk+j+1, \dots, vk+j+k$, divisible by 2^α. Among integers $vk+j+1, \dots, vk+j+k$, it is also clear that there are at most two integers, say $vk+j+i_1$ and $vk+j+i_2$, which are divisible by 2^α. If $i_1 \ne i_2$, then only one of them, say $vk+j+i_1$, is divisible by $2^{\alpha+1}$. By assumption $vk+j+i_1 \ne (v+1)k$, therefore $\chi(vk+j+i_1) \ne 0$, and by Proposition 1, we have that $T(v, j, \chi) \ne 0$. If $i_1 = i_2$, again by assumption and Proposition 1, $T(v, j, \chi) \ne 0$. □

Now, we are ready to derive Theorem 1:

Theorem 1. *Let χ be a real primitive character modulo a positive odd prime integer k, $k \ge 3$ and v a nonnegative integer. Then*

$$T(v, j, \chi) \ne 0 \text{ for } j = 0,1,2,\dots, k-1.$$

Proof. The case $v = 0$ has been proved in Proposition 1 of [9]. For $v \ne 0$, we divide the argument into two cases:

Case 1. $k > 100$.

By Lemma 4, it is enough to discuss the case $2^\alpha \mid (v+1)k$, where α is the integer such that $2^\alpha < k < 2^{\alpha+1}$.

For any fixed positive integer v such that $2^\alpha \mid (v+1)k$ and any fixed integer j ($j = 0,1,2,\dots, k-1$), by Lemma 2, there exists a prime p, $p > \frac{3}{2}k$ such that $p \mid (vk+j+i_0)$ for some integer i_0 in the closed interval $[1, k]$. If $j + i_0 \ne k$, by Proposition 1, $T(v, j, \chi) \ne 0$. If $j + i_0 = k$, then either $i_0 \in \left[1, \left[\frac{k}{2}\right]\right]$ or $i_0 \in \left[\left[\frac{k}{2}\right]+1, k\right]$. If $i_0 \in \left[\left[\frac{k}{2}\right]+1, k\right]$, by Proposition 2, there exists a prime q, $q > 2\left[\frac{k}{2}\right]$ such that $q \mid (vk+j+i_1)$ for some integer i_1 in the closed interval $\left[1, \left[\frac{k}{2}\right]\right]$. (Note. In the case $j + i_0 = k$, we have that $vk \ge 2^\alpha pk - k = k(2^\alpha p - 1) > k^{\frac{100}{38}}$ for $k > 100$.) Since $q > k$, we know that $q \nmid (vk+j+i)$ for any integer i in $[1, k]$ and $i \ne i_1$. By Proposition 1, $T(v, j, \chi) \ne 0$. For the case $i_0 \in \left[1, \left[\frac{k}{2}\right]\right]$, the similar argument implies that $T(v, j, \chi) \ne 0$.

Case 2. $3 \le k < 100$.

By applying Proposition 1, Lemma 2, Lemma 4 and the results of Lehmer [8], we have that $T(v, j, \chi) \ne 0$ for any positive integer v and integer j, $0 \le j \le k-1$. □

3. Estimating $L(1, \chi)$. In this section, we use Abel's identity and Pólya's inequality to derive the inequalities (1.2) and in the next section, as an application, we use inequalities (1.2) to give a class number formula for the real quadratic fields.

We begin by recalling the results of Abel and Pólya.

Lemma 5. *For any arithmetical function $a(n)$ let*

$$A(x) = \sum_{n \le x} a(n),$$

where $A(x) = 0$ if $x < 1$. Assume f has a continuous derivative on the interval $[y, x]$, where $0 < y < x$. Then

$$\sum_{y<n\le x} a(n) f(n) = A(x) f(x) - A(y) f(y) - \int_y^x A(t) f'(t)\, dt.$$

Lemma 6. *Let χ be a primitive character modulo integer k, $S = \sum_{n<B} \chi(n)$. Then*

$$|S| < \sqrt{k} \ln k.$$

The proofs of Lemma 5 and Lemma 6 can be found in [1, pp. 77] and in [1, pp. 173], respectively. We remark that the integer k in Lemma 6 can be either a prime or a composite integer.

From the definition of Kronecker character we know that $\chi(n) = \chi(-n) sgn(d)$, where d is the fundamental discriminant equal to k or $-k$ (cf. [2, page 292]). If both k and $-k$ are fundamental discriminants (which happens if and only if $k = 8k'$, where k' is odd and squarefree) there are two real primitive characters (Kronecker character)(mod k), otherwise only one. Clearly, we have that $\chi(-1) = 1$ if and only if $d > 0$. In this section and the next section we restrict ourselves to the case $d = k$. Fix such an integer k, let χ be a real primitive character attached to the real quadratic field $\boldsymbol{Q}(\sqrt{k})$ with $\chi(-1) = 1$.

Let $A(x) = \sum_{n \le x} \chi(n)$ and $f(x) = \frac{1}{x}$, then, applying the properties $\chi(-1) = 1$ and $\sum_{n=1}^{k} \chi(n) = 0$, we have $A\left(\frac{k}{2}\right) = A\left(\left[\frac{k}{2}\right]\right) = A(k) = 0$. By Theorem B mentioned in section 1 and

Abel's identity (Lemma 5), we easily have the following theorem.

Theorem 2. $\sum_{n=1}^{[r]} \frac{\chi(n)}{n} - \frac{A(r)}{r} + \int_r^k \frac{A(t)}{t^2} dt < L(1, \chi) < \sum_{n=1}^{[r]} \frac{\chi(n)}{n} - \frac{A(r)}{r} + \int_r^{[\frac{k}{2}]} \frac{A(t)}{t^2} dt$, *where* $1 \leq r \leq \left[\frac{k}{2}\right]$.

Remark. It might be possible that there exist integers k and small integer $r = r(k)$ such that $A(t)$ is bounded by small number for t in the interval $\left[r, \left[\frac{k}{2}\right]\right]$ or even better situation may occur for t in the interval $[r, k]$. In those cases, the finite sum $\sum_{n=1}^{r} \frac{\chi(n)}{n}$ can be used to estimate $L(1, \chi)$.

For the case $r = \frac{k}{2}$, we obtain, as a corollary of Theorem 2, the inequalities (1.2):

Theorem 3. $\sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n} - \frac{\ln k}{\sqrt{k}} < L(1, \chi) < \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}$.

Proof. From Theorem B, we have

$$\sum_{n=1}^{k} \frac{\chi(n)}{n} < L(1, \chi) < \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}.$$

Write $\sum_{n=1}^{k} \frac{\chi(n)}{n} = \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n} + \sum_{n=[\frac{k}{2}]+1}^{k} \frac{\chi(n)}{n}$. By applying Abel's identity, we have

$$\sum_{n=[\frac{k}{2}]+1}^{k} \frac{\chi(n)}{n} = \frac{A(k)}{k} - \frac{A\left(\frac{k}{2}\right)}{\frac{k}{2}} - \int_{\frac{k}{2}}^{k} A(t) f'(t) dt$$

$$= \int_{\frac{k}{2}}^{k} \frac{A(t)}{t^2} dt,$$

where $A(x) = \sum_{n \leq x} \chi(n)$ and $f(x) = \frac{1}{x}$ for $x > 0$. Now applying Pólya's inequality, we have

$$\left| \sum_{n=[\frac{k}{2}]+1}^{k} \frac{\chi(n)}{n} \right| \leq \int_{\frac{k}{2}}^{k} \frac{|A(t)|}{t^2} dt < \sqrt{k} \ln k \int_{\frac{k}{2}}^{k} \frac{1}{t^2} dt = \frac{\ln k}{\sqrt{k}}.$$

Therefore, we obtain the desired inequalities

$$\sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n} - \frac{\ln k}{\sqrt{k}} < \sum_{n=1}^{k} \frac{\chi(n)}{n} < L(1, \chi) < \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}.$$

□

Remark. Applying Theorem 1 of [15], we know that $L(1, \chi) > \frac{1}{40k^{1/4}}$ with one possible exception. Since $\frac{1}{40k^{1/4}} > \frac{\ln k}{\sqrt{k}}$ for large k and $\sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n} > L(1, \chi)$, it is quite sure that $\sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n} - \frac{\ln k}{\sqrt{k}} > 0$ for large integer k. As an example, we consider $k = 17$, we have that $\sum_{n=1}^{8} \frac{\chi(n)}{n} - \frac{\ln 17}{\sqrt{17}} \approx 0.344987688$ is better than $\frac{1}{40\sqrt[4]{17}} \approx 0.012311976$.

4. Class number formula. Dirichlet's class number formula asserts that

$$h = \frac{\sqrt{k}}{2\ln \varepsilon} L(1, \chi),$$

where k is the fundamental discriminant, h is the class number, and $\varepsilon (> 1)$ is the fundamental unit of $\boldsymbol{Q}(\sqrt{k})$. Before deriving a new class number formula for the real quadratic fields, we recall a well-known result:

Lemma 7. *If the discriminant of a quadratic field contains only one prime factor, then the class number of the field is odd.*

The proof can be found in [4, pp. 187].

As an application of Theorem 3, we have the following inequalities.

Theorem 4.

$$\frac{\sqrt{k}}{2\ln \varepsilon} \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n} - \frac{\ln k}{2\ln \varepsilon} < h = \frac{\sqrt{k}}{2\ln \varepsilon} L(1, \chi) < \frac{\sqrt{k}}{2\ln \varepsilon} \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}.$$

Corollary 1. *If* $k \equiv 1 \ (mod\ 4)$ *is a prime, then the class number* $h = 1$ *if and only if* $\sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n} \leq \frac{6\ln \varepsilon}{\sqrt{k}}$.

Proof. By Lemma 7, the class number h is odd. If $\frac{\sqrt{k}}{2\ln \varepsilon} \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n} \leq 3$, then, by Theorem 4, $h = 1$. Again, by using Theorem 4, we have

$$\frac{\sqrt{k}}{2\ln \varepsilon} \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n} < h + \frac{\ln k}{2\ln \varepsilon}.$$

If $h = 1$, we see that

$$\sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n} < \frac{2\ln \varepsilon}{\sqrt{k}} + \frac{\ln k}{\sqrt{k}}$$
$$= \frac{2\ln \varepsilon}{\sqrt{k}} + \frac{2\ln \frac{\sqrt{k}}{2}}{\sqrt{k}} + \frac{2\ln 2}{\sqrt{k}}$$
$$\leq \frac{2\ln \varepsilon}{\sqrt{k}} + \frac{2\ln \varepsilon}{\sqrt{k}} + \frac{2\ln \varepsilon}{\sqrt{k}} = \frac{6\ln \varepsilon}{\sqrt{k}}.$$
□

Corollary 2. *If positive integer k is not of the form $m^2 + 4$ ($m \in \boldsymbol{N}$), then*
$$h = \left[\frac{\sqrt{k}}{2\ln \varepsilon} \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}\right].$$

Proof. Let $\varepsilon = \dfrac{a + b\sqrt{k}}{2}$ (> 1) be the fundamental unit of $\boldsymbol{Q}(\sqrt{k})$. From the properties $|\varepsilon\bar{\varepsilon}| = 1$ and $\varepsilon > 1$, we know $b > 0$.

We divide the argument into two cases:

Case 1. $b > 1$.

Clearly, $\varepsilon^2 > k$. Thus $\dfrac{\ln k}{2\ln \varepsilon} < 1$, which implies, by Theorem 4,
$$h = \left[\frac{\sqrt{k}}{2\ln \varepsilon} \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}\right].$$

Case 2. $b = 1$.

Since $|\varepsilon\bar{\varepsilon}| = 1$ and k, by assumption, is not of the form $m^2 + 4$ ($m \in \boldsymbol{N}$), we have $a^2 - k = 4$. Because $\varepsilon > 1$, so $a = \sqrt{k + 4}$. We have $\varepsilon = \dfrac{\sqrt{k+4} + \sqrt{k}}{2} > \sqrt{k}$. Hence $\dfrac{\ln k}{2\ln \varepsilon} < 1$, which implies
$$h = \left[\frac{\sqrt{k}}{2\ln \varepsilon} \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}\right].$$
□

Corollary 3. *If $k = m^2 + 4$ ($m \in \boldsymbol{N}$ and $m \geq 3$), then*
$$h = \left[\frac{\sqrt{k}}{2\ln \varepsilon} \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}\right] - i,$$
where $i = 0$ or $i = 1$ depends on whether $\left[\frac{\sqrt{k}}{2\ln \varepsilon} \sum_{n=1}^{[\frac{k}{2}]} \frac{\chi(n)}{n}\right] + E(k)$ is an even integer or an odd integer, where $E(k)$ is 1 or 0 depending on whether k is a prime or not a prime.

Proof. For simplicity, let $A = \dfrac{\sqrt{k}}{2\ln \varepsilon} \sum_{n=1}^{[\frac{k}{2}]} \dfrac{\chi(n)}{n}$. Since the fundamental unit $\varepsilon = \dfrac{m + \sqrt{m^2 + 4}}{2}$, we easily have $\varepsilon^3 > k$ which gives $\dfrac{3}{2} > \dfrac{\ln k}{2\ln \varepsilon}$. By Theorem 4, $A - \dfrac{3}{2} < A - \dfrac{\ln k}{2\ln \varepsilon} < h < A$. Therefore, there are at most two distinct integers contained in the interval $\left[A - \frac{3}{2}, A\right]$.

We divide the argument into two cases:

Case 1. k is prime.

By Lemma 7, the class number h is odd. Hence $h = [A] - i$, where $i = 0$ or 1 depends on the integer $[A]$ is odd or even.

Case 2. k is not a prime.

By the genus theory of quadratic number fields, the class number $h = h^+ = 2^{t-1}h^*$ (for the case $\varepsilon\bar{\varepsilon} = -1$), where h^+ is the class number of $\boldsymbol{Q}(\sqrt{k})$ in the narrow sense, h^* is the number of classes in a genus, and t is the number of distinct prime factors of k. The restriction on k gives that the class number h is even. Hence $h = [A] - i$, where $i = 0$ or 1 depends on the integer $[A]$ is even or odd. □

Combining Theorem 4 and the class number formula of Ono [11], we can get the following interesting inequalities without involving the class number h and the fundamental unit ε.

Corollary 4. *Let $p \equiv 1 \pmod 4$ be a prime. Then*
$$\frac{\sqrt{p}}{2} \sum_{n=1}^{[\frac{p}{2}]} \frac{\chi(n)}{n} > \ln\left(\frac{2}{\sqrt{p}} \sum_{n=1}^{N-1} d_n + \frac{d_N}{\sqrt{p}}\right)$$
$$> \frac{\sqrt{p}}{2} \sum_{n=1}^{[\frac{p}{2}]} \frac{\chi(n)}{n} - \frac{\ln p}{2},$$
where $N = \dfrac{p-1}{4}$, $d_0 = 1$ and $2nd_n = \sum_{v=1}^{n}\left(1 + \left(\frac{v}{p}\right)\sqrt{p}\right)d_{n-v}$, $1 \leq n \leq N$. (Here $\left(\frac{x}{y}\right)$ denotes the Legendre symbol.)

Proof. By [11], we have
$$h\ln \varepsilon = \ln\left(\frac{2}{\sqrt{p}} \sum_{n=1}^{N-1} d_n + \frac{d_N}{\sqrt{p}}\right).$$
On the other hand, by Theorem 4, we have
$$\frac{\sqrt{p}}{2\ln \varepsilon} \sum_{n=1}^{[\frac{p}{2}]} \frac{\chi(n)}{n} > h > \frac{\sqrt{p}}{2\ln \varepsilon} \sum_{n=1}^{[\frac{p}{2}]} \frac{\chi(n)}{n} - \frac{\ln p}{2\ln \varepsilon},$$
hence Corollary follows. □

References

[1] T. M. Apostol: Introduction to Analytic Number Theory. Springer-Verlag, New York, Belin, Heidelberg, Tokyo (1976).
[2] R. Ayoub: An Introduction to the Analytic Theory of Numbers. Mathematical surveys, no. 10, Amer. Math. Soc., Providence (1963).
[3] D. A. Burgess: Estimating $L_\chi(1)$, Norske Vid. Selsk. Forh. (Trondheim), **39** (1966); 101–108 (1967).
[4] H. Cohn: Advanced Number Theory. Dover, New York (1980).
[5] H. Davenport: On the series for $L(1)$. J. London Math. Soc., **24**, 229–233 (1949).
[6] P. Erdös: A theorem of Sylvester and Schur. J. London Math. Soc., **9**, 282–288 (1934).
[7] D. Hanson: On a theorem of Sylvester and Schur. Canad. Math. Bull., **16**, 195–199 (1973).
[8] D. H. Lehmer: The prime factors of consecutive integers. Amer. Math. Monthly 72, no. 2, part II, 19–20 (1965).
[9] M.-G. Leu and W.-C. Li: On the series for $L(1, \chi)$ (to appear in Nagoya Mathematical Journal).
[10] S. Louboutin: Majorations explicites de $|L(1, \chi)|$. C. R. Acad. Sci. Paris, Sér. I Math., **316**, 11–14 (1993).
[11] T. Ono: A deformation of Dirichlet's class number formula. Algebraic Analysis, **2**, 659–666 (1988).
[12] J. Pintz: Elementary methods in the theory of L-functions. VII. Upper bound for $L(1, \chi)$, Acta. Arith., **32**, 397–406 (1977).
[13] C. L. Siegel: Über die Classenzahl quadratischer Zahlkörper. Acta. Arith., **1**, 83–86 (1935).
[14] P. J. Stephens: Optimizing the size of $L(1, \chi)$. Proc. London Math. Soc., **24**, 1–14 (1972).
[15] T. Tatuzawa: On a theorem of Siegel. Japan. J. Math., **21**, 163–178 (1951).