The Relative Class Number of Certain Imaginary Abelian Number Fields of Odd Conductors*)

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1. Introduction. The class number of an imaginary abelian number field is divisible by that of its maximal real subfield and the quotient is called the relative class number of it.

Let p be an odd prime number. For a rational integer a prime to p, we denote by $R(a)$ the least positive residue of a modulo p. Then Maillet's determinant D_p is defined by

$$D_p = |R(ab')|_{1\leq a,b\leq r},$$

where $r = (p-1)/2$ and b' is a rational integer which satisfies $bb' \equiv 1 \pmod{p}$.

Let Q and ζ be the field of rational numbers and a primitive p-th root of unity, respectively. Carlitz and Olson [1] proved that D_p is a multiple of the relative class number h_p^- of the p-th cyclotomic number field $Q(\zeta)$. This result has been generalized to more general imaginary abelian number fields [8], [11], [12], [15], [16].

On the other hand, recently Hazama [10] showed that the determinant of the Demjanenko matrix provides the formula for h_p^-. The Demjanenko matrix is defined by

$$(C(ab))_{1\leq a,b\leq r};$$

herein for a rational integer a prime to p $C(a) = 1$ if $R(a) < p/2$, and $C(a) = 0$ if $R(a) > p/2$. Hazama's formula has been also generalized to more general imaginary abelian number fields of odd conductors [2], [7], [9], [13].

In the previous papers [3], [4] we investigated the Stickelberger ideal of quadratic extensions of $Q(\zeta)$ and obtained a formula for the relative class number of such imaginary abelian number fields. Our formula is expressed as a product of two determinants of degree r. In this paper we consider the Demjanenko matrix and show a new relative class number formula expressed as a product of two determinants of degree r.

*) Dedicated to Professor Katsumi Shiratani on his 63rd birthday.

2. Statement of the theorem. Let m be a square-free rational integer such that $m \equiv 1 \pmod{4}$, and d its absolute value. We consider the quadratic extension $K = Q(\zeta, \sqrt{m})$ of $Q(\zeta)$ obtained by adjoining $\sqrt{m}$. We may assume without loss of generality that m is prime to p. Let Z be the ring of rational integers and N the subgroup of the multiplicative group $(Z/dZ)^\times$ corresponding to $Q(\sqrt{m})$ by Galois theory; then the Galois group G of K/Q is isomorphic to the direct product of the multiplicative group $(Z/pZ)^\times$ and the quotient group $(Z/dZ)^\times/N$.

For each $1 \leq a \leq p-1$ we choose a rational integer a^* prime to dp so that $a^* \equiv a \pmod{p}$ and $1^*, 2^*, \dots, (p-1)^*$ form a complete system of representatives for $G/\{\pm 1\}$; then we see $(p-a)^* \not\equiv -a^* \pmod{N}$ and we may take $1^* = 1$.

Now, for a rational integer a prime to dp we denote by $c_a^{(K)}$ and $c'^{(K)}_a$ respectively the number of $1 \leq x \leq (dp-1)/2$ such that $x \equiv a \pmod{p}$ and $x \equiv a \pmod{N}$, and that of $(dp+1)/2 \leq x \leq dp-1$ such that $x \equiv a \pmod{p}$ and $x \equiv a \pmod{N}$. We define the Demjanenko matrix for K by

$$(c_{a^*b^*}^{(K)} - c'^{(K)}_{b^*})_{1\leq a,b\leq p-1}$$

[2], and denote its determinant by $H^{(K)}$.

Let X be the group of the primitive Dirichlet characters associated with $Q(\sqrt{m})$, and further $\chi_0 \in X$ the principal character of conductor d. For any $\chi \in X$ and a rational integer a prime to p, let

$$C_a(\chi) = \sum_{x=1}^{(dp-1)/2}{}^{(a)} \chi(x)$$

and

$$C'_a(\chi) = \sum_{x=(dp+1)/2}^{dp-1}{}^{(a)} \chi(x),$$

where (a) indicates that x runs through rational integers in the assigned interval which are prime to dp and congruent to a modulo p. We then define a determinant $H_p(\chi)$ of degree r by

$$H_p(\chi)=\begin{cases}|C_{ab}(\chi_0)-C'_b(\chi_0)|_{1\le a,b\le r} & \text{if } \chi=\chi_0,\\ |C_{ab}(\chi)|_{1\le a,b\le r} & \text{if } \chi\ne\chi_0.\end{cases}$$

Let ψ be a primitive Dirichlet character of degree $p-1$ associated with $Q(\zeta)$. For $1\le i\le p-1$ let

$$B_{1,\psi^i}=\frac{1}{p}\sum_{x=1}^{p-1}\psi^i(x)x$$

and

$$B_{1,\psi^i\chi}=\frac{1}{dp}\sum_{x=1}^{dp-1}(\psi^i\chi)(x)x,\ \chi\in X-\{\chi_0\}$$

be the generalized Bernoulli numbers belonging to ψ^i amd $\psi^i\chi$, respectively. For a prime number l let f_l be the order of l modulo p.

Theorem 1. *With the notation above we have the following*:

(1) $H^{(K)}=\prod_{\chi\in X}H_p(\chi)$.

(2) $H_p(\chi_0)\ne 0$ *if and only if* $f_l\equiv 0 \pmod 2$ *for any prime divisor* l *of* d, *in which case*

$$|H_p(\chi_0)|=2\prod_{l|d}2^{(p-1)/f_l}\prod_{i=1}^{r}\left|(2-\psi^{2i-1}(2))\frac{1}{2}B_{1,\psi^{2i-1}}\right|,$$

where $\prod_{l|d}$ *means the product taken over all prime divisors* l *of* d.

(3) *Let* $\chi\in X-\{\chi_0\}$. $H_p(\chi)\ne 0$ *if and only if* $m>0$ *or* $\chi(p)=-1$, *in which case*

$$|H_p(\chi)|=\begin{cases}\prod_{i=1}^{r}\left|(2-(\psi^{2i-1}\chi)(2))\frac{1}{2}B_{1,\psi^{2i-1}\chi}\right| & \textit{if } m>0,\\ 2\prod_{i=1}^{r}\left|(2-(\psi^{2i}\chi)(2))\frac{1}{2}B_{1,\psi^{2i}\chi}\right| & \textit{if } m<0.\end{cases}$$

Remark. It is well known that $\chi(2)=(-1)^{(m-1)/4}$ for $\chi\in X-\{\chi_0\}$, and further it is easy to see that the following holds:

$$\prod_{i=1}^{r}(2-\psi^{2i-1}(2))=\begin{cases}(2^{f_2/2}+1)^{(p-1)/f_2} & \text{if } f_2\equiv 0 \pmod 2,\\ (2^{f_2}-1)^{(p-1)/2f_2} & \text{if } f_2\equiv 1 \pmod 2,\end{cases}$$

$$\prod_{i=1}^{r}(2-\psi^{2i}(2))=\begin{cases}(2^{f_2/2}-1)^{(p-1)/f_2} & \text{if } f_2\equiv 0 \pmod 2,\\ (2^{f_2}-1)^{(p-1)/2f_2} & \text{if } f_2\equiv 1 \pmod 2,\end{cases}$$

$$\prod_{i=1}^{r}(2+\psi^{2i-1}(2))=\begin{cases}(2^{f_2/2}+1)^{(p-1)/f_2} & \text{if } f_2\equiv 0 \pmod 4,\\ (2^{f_2/2}-1)^{(p-1)/f_2} & \text{if } f_2\equiv 2 \pmod 4,\\ (2^{f_2}+1)^{(p-1)/2f_2} & \text{if } f_2\equiv 1 \pmod 2,\end{cases}$$

$$\prod_{i=1}^{r}(2+\psi^{2i}(2))=\begin{cases}(2^{f_2/2}-1)^{(p-1)/f_2} & \text{if } f_2\equiv 0 \pmod 4,\\ (2^{f_2/2}+1)^{(p-1)/f_2} & \text{if } f_2\equiv 2 \pmod 4,\\ (2^{f_2}+1)^{(p-1)/2f_2} & \text{if } f_2\equiv 1 \pmod 2.\end{cases}$$

Let Q_K be the unit index of K (cf. [17]). When $m>0$, that $Q_K=1$ follows from the fact that a prime ideal dividing p in $Q(\zeta+\zeta^{-1},\sqrt{m})$ is ramified in K. Then the following corollary is an immediate consequence of the analytic formula for the relative class number h_K^- of K and the above theorem together with the remark (cf. [5]).

Corollary. *If* $\prod_{\chi\in X}H_p(\chi)\ne 0$, *then*

$$\left|\prod_{\chi\in X}H_p(\chi)\right|=\begin{cases}\dfrac{F}{p}\prod_{l|d}2^{(p-1)/f_l}h_K^- & \textit{if } m>0,\\ \dfrac{4F}{Q_Kw_K}\prod_{l|d}2^{(p-1)/f_l}h_K^- & \textit{if } m<0.\end{cases}$$

Herein $w_K=6p$ *or* $2p$ *according as* $m=-3$ *or not, and when* $m>0$

$$F=\begin{cases}(2^{f_2/2}+1)^{2(p-1)/f_2} & \\ \quad \textit{if } m\equiv 1 \pmod 8,\ f_2\equiv 0 \pmod 2, & \\ \quad \textit{or } m\equiv 5 \pmod 8,\ f_2\equiv 0 \pmod 4, & \\ (2^{f_2}-1)^{(p-1)/f_2} & \\ \quad \textit{if } m\equiv 1 \pmod 8,\ f_2\equiv 1 \pmod 2, & \\ \quad \textit{or } m\equiv 5 \pmod 8,\ f_2\equiv 2 \pmod 4, & \\ (2^{2f_2}-1)^{(p-1)/2f_2} & \\ \quad \textit{if } m\equiv 5 \pmod 8,\ f_2\equiv 1 \pmod 2, & \end{cases}$$

and when $m<0$

$$F=\begin{cases}(2^{f_2/2}+1)^{2(p-1)/f_2} & \\ \quad \textit{if } m\equiv 5 \pmod 8,\ f_2\equiv 2 \pmod 4, & \\ (2^{f_2}-1)^{(p-1)/f_2} & \\ \quad \textit{if } m\equiv 1 \pmod 8\ \textit{or } f_2\equiv 0 \pmod 4, & \\ (2^{2f_2}-1)^{(p-1)/2f_2} & \\ \quad \textit{if } m\equiv 5 \pmod 8,\ f_2\equiv 1 \pmod 2. & \end{cases}$$

3. Proof of the theorem. In this section we give the proof of Theorem 1.

Proof of Theorem 1. (1) Since

$$c_a^{(K)}+c_a'^{(K)}=\frac{1}{2}\varphi(d),$$

where φ is the Euler function, for any rational integer a prime to dp,

$$\begin{aligned}&c_{a^*b^*}^{(K)}-c_{b^*}'^{(K)}\\&=\left(c_{a^*b^*}^{(K)}-\frac{1}{4}\varphi(d)\right)-\left(c_{b^*}'^{(K)}-\frac{1}{4}\varphi(d)\right)\\&=\frac{1}{2}\left(c_{a^*b^*}^{(K)}-c_{a^*b^*}'^{(K)}\right)+\frac{1}{2}\left(c_{b^*}^{(K)}-c_{b^*}'^{(K)}\right).\end{aligned}$$

Hence we have

$$H^{(K)} = | c^{(K)}_{a^*b^*} - c'^{(K)}_{b^*} |_{1\le a,b\le p-1}$$
$$= 2\left| \frac{1}{2}(c^{(K)}_{a^*b^*} - c'^{(K)}_{a^*b^*}) \right|_{1\le a,b\le p-1}.$$

Since $(p-a)^* \not\equiv -a^* \pmod N$ and $(p-b)^* \not\equiv -b^* \pmod N$, we have $a^*(p-b)^* \equiv (p-a)^*b^* \pmod N$ and $(p-a)^*(p-b)^* \equiv a^*b^* \pmod N$, which implies

$$c^{(K)}_{a^*(p-b)^*} = c^{(K)}_{(p-a)^*b^*}, \quad c'^{(K)}_{a^*(p-b)^*} = c'^{(K)}_{(p-a)^*b^*}$$

and

$$c^{(K)}_{(p-a)^*(p-b)^*} = c^{(K)}_{a^*b^*}, \quad c'^{(K)}_{(p-a)^*(p-b)^*} = c'^{(K)}_{a^*b^*}.$$

Thus we see

$$H^{(K)} = 2\begin{vmatrix} A & B \\ B & A \end{vmatrix},$$

where

$$A = \left(\frac{1}{2}(c^{(K)}_{a^*b^*} - c'^{(K)}_{a^*b^*})\right)_{1\le a,b\le r},$$
$$B = \left(\frac{1}{2}(c^{(K)}_{a^*(p-b)^*} - c'^{(K)}_{a^*(p-b)^*})\right)_{1\le a,b\le r}.$$

Further it is easy to see that for a rational integer x the two conditions

$$\frac{1}{2}(dp+1) \le x \le dp-1,\ x \equiv -ab \pmod p,$$
$$x \equiv a^*(p-b)^* \pmod N$$

and

$$1 \le dp - x \le \frac{1}{2}(dp-1),\ dp - x \equiv ab \pmod p,$$
$$dp - x \not\equiv a^*b^* \pmod N$$

are equivalent. Hence we have

$$c^{(K)}_{a^*b^*} + c'^{(K)}_{a^*(p-b)^*} = C_{ab}(\chi_0)$$

and

$$c^{(K)}_{a^*b^*} - c'^{(K)}_{a^*(p-b)^*} = \chi(a^*b^*)C_{ab}(\chi), \quad \chi \in X - \{\chi_0\}.$$

Similarly we have

$$c'^{(K)}_{a^*b^*} + c^{(K)}_{a^*(p-b)^*} = C'_{ab}(\chi_0)$$

and

$$c'^{(K)}_{a^*b^*} - c^{(K)}_{a^*(p-b)^*} = \chi(a^*b^*)C'_{ab}(\chi), \quad \chi \in X - \{\chi_0\}.$$

These imply

$$\frac{1}{2}(c^{(K)}_{a^*b^*} - c'^{(K)}_{a^*b^*}) - \frac{1}{2}(c^{(K)}_{a^*(p-b)^*} - c'^{(K)}_{a^*(p-b)^*})$$
$$= \frac{1}{2}(C_{ab}(\chi_0) - C'_{ab}(\chi_0))$$

and

$$\frac{1}{2}(c^{(K)}_{a^*b^*} - c'^{(K)}_{a^*b^*}) + \frac{1}{2}(c^{(K)}_{a^*(p-b)^*} - c'^{(K)}_{a^*(p-b)^*})$$
$$= \frac{1}{2}\chi(a^*b^*)(C_{ab}(\chi) - C'_{ab}(\chi)),\ \chi \in X - \{\chi_0\}.$$

Therefore an easy calculation on rows and columns of determinants shows

$$H^{(K)} = 2\left| \frac{1}{2}(C_{ab}(\chi_0) - C'_{ab}(\chi_0)) \right|_{1\le a,b\le r}$$
$$\cdot \left| \frac{1}{2}\chi(a^*b^*)(C_{ab}(\chi) - C'_{ab}(\chi)) \right|_{1\le a,b\le r}$$
$$= 2\left| \frac{1}{2}(C_{ab}(\chi_0) - C'_{ab}(\chi_0)) \right|_{1\le a,b\le r}$$
$$\cdot \left| \frac{1}{2}(C_{ab}(\chi) - C'_{ab}(\chi)) \right|_{1\le a,b\le r},$$

where $\chi \in X - \{\chi_0\}$. Since

$$C_a(\chi) + C'_a(\chi) = \begin{cases} \varphi(d) & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \in X - \{\chi_0\} \end{cases}$$

for any rational integer a prime to p, we have

$$\frac{1}{2}(C_{ab}(\chi_0) - C'_{ab}(\chi_0)) + \frac{1}{2}(C_b(\chi_0) - C'_b(\chi_0))$$
$$= C_{ab}(\chi_0) - C'_b(\chi_0)$$

and

$$\frac{1}{2}(C_{ab}(\chi) - C'_{ab}(\chi)) = C_{ab}(\chi),\ \chi \in X - \{\chi_0\}.$$

Hence we obtain

$$H^{(K)} = \prod_{\chi \in X} H_p(\chi).$$

(2) From the above we see

$$H_p(\chi_0) = \frac{1}{2^{r-1}} | C_{ab}(\chi_0) - C'_{ab}(\chi_0) |_{1\le a,b\le r}$$
$$= \pm\frac{1}{2^{r-1}} | \psi(ab')(C_{ab'}(\chi_0) - C'_{ab'}(\chi_0)) |_{1\le a,b\le r}.$$

It can be easily seen that

$$\psi(a)(C_a(\chi_0) - C'_a(\chi_0))$$

is a function on $(Z/pZ)^\times/\{\pm 1\}$. Hence by the formula for abelian group determinant (cf. [17]) we obtain

$$| \psi(ab')(C_{ab'}(\chi_0) - C'_{ab'}(\chi_0)) |_{1\le a,b\le r}$$
$$= \prod_{i=1}^{r} \sum_{a=1}^{r} \psi^{2i-1}(a)(C_a(\chi_0) - C'_a(\chi_0))$$
$$= \prod_{i=1}^{r} (\sum_{a=1}^{r} \psi^{2i-1}(a)C_a(\chi_0) + \sum_{a=r+1}^{p-1} \psi^{2i-1}(a)C_a(\chi_0))$$
$$= \prod_{i=1}^{r} \sum_{a=1}^{p-1} \sideset{}{^{(a)}}\sum_{x=1}^{(dp-1)/2} \psi^{2i-1}(x)\chi_0(x)$$
$$= \prod_{i=1}^{r} \sum_{\substack{1\le x\le (dp-1)/2 \\ (x,dp)=1}} \psi^{2i-1}(x).$$

It follows from an easy calculation that

$$\sum_{\substack{1\le x\le (dp-1)/2 \\ (x,dp)=1}} \psi^{2i-1}(x) = -\frac{1}{dp}(2 - \bar{\psi}^{2i-1}(2)) \sum_{\substack{1\le x\le dp-1 \\ (x,pd)=1}} \psi^{2i-1}(x)x$$

(cf. [6]). Further it is well known that

$$\frac{1}{dp} \sum_{\substack{1\le x\le dp-1 \\ (x,dp)=1}} \psi^{2i-1}(x)x = \prod_{l|d} (1 - \psi^{2i-1}(l))B_{1,\psi^{2i-1}}$$

(cf. [4]). Since

$$\prod_{i=1}^{r} (1 - \psi^{2i-1}(l)) = \begin{cases} 2^{(p-1)/f_l} & \text{if } f_l \equiv 0 \pmod 2, \\ 0 & \text{if } f_l \equiv 1 \pmod 2, \end{cases}$$

our assertion is obtained.

(3) The assertion of the third part is also obtained by the way similar to the above.

4. The case where $m = -3$. In what follows we assume $m = -3$ and so $d = 3$. For a rational integer a prime to p, we denote by $R'(a)$ a rational integer which satisfies $R'(a) \equiv \pm a \pmod p$ and $1 \leq R'(a) \leq r$. Let

$$C^{(3)}(a) = \begin{cases} 1 & \text{if } 1 \leq R(a) \leq r,\ R'(a) \equiv p \pmod 3, \\ -1 & \text{if } r+1 \leq R(a) \leq p-1,\ R'(a) \equiv p \pmod 3, \\ 0 & \text{if } R'(a) \not\equiv p \pmod 3 \end{cases}$$

and put

$$H_p^{(3)} = |\, C^{(3)}(ab)\,|_{1 \leq a,b \leq r}.$$

Further let

$$G^{(3)}(a) = \begin{cases} 1 & \text{if } R'(a) \equiv p-1 \pmod 3, \\ -1 & \text{if } R'(a) \equiv p+1 \pmod 3, \\ 0 & \text{if } R'(a) \equiv p \pmod 3 \end{cases}$$

and put

$$G_p^{(3)} = |\, G^{(3)}(ab)\,|_{1 \leq a,b \leq r}.$$

Then it is easy to see that

$$C^{(3)}(a) = \frac{1}{2}(C_a(\chi_0) - C'_a(\chi_0))$$

and

$$G^{(3)}(a) = C_a(\chi), \quad \chi \in X - \{\chi_0\}.$$

Let $h^-_{p,-3}$ denote the relative class number of $Q(\zeta, \sqrt{-3})$. Noting that $Q_K = 2$ for $K = Q(\zeta, \sqrt{-3})$ and that when $p \equiv 2 \pmod 3$, $f_3 \equiv 0 \pmod 2$ if and only if $p \equiv 5 \pmod{12}$, the following theorem is established immediately from Theorem 1 and its corollary.

Theorem 2. *With the notation above we have the following*:

$$|\, H_p^{(3)} \,| = \begin{cases} 2^{(p-1)/f_3} \prod_{i=1}^{r} |\, (2 - \psi^{2i-1}(2)) \frac{1}{2} B_{1,\psi^{2i-1}} \,| & \\ & \textit{if } f_3 \equiv 0 \pmod 2, \\ 0 & \textit{if } f_3 \equiv 1 \pmod 2, \end{cases}$$

$$|\, G_p^{(3)} \,| = \begin{cases} 2 \prod_{i=1}^{r} |\, (2 + \psi^{2i}(2)) \frac{1}{2} B_{1,\psi^{2i}\chi} \,| & \\ & \textit{if } p \equiv 2 \pmod 3, \\ 0 & \textit{if } p \equiv 1 \pmod 3, \end{cases}$$

$$|\, H_p^{(3)} \cdot G_p^{(3)} \,| = \begin{cases} \frac{F}{6p} 2^{(p-1)/f_3} h^-_{p,-3} & \\ & \textit{if } p \equiv 5 \pmod{12}, \\ 0 & \textit{if } p \not\equiv 5 \pmod{12}, \end{cases}$$

where $\chi \in X - \{\chi_0\}$ *and*

$$F = \begin{cases} (2^{f_2} - 1)^{(p-1)/f_2} & \textit{if } f_2 \equiv 0 \pmod 4, \\ (2^{f_2/2} + 1)^{2(p-1)/f_2} & \textit{if } f_2 \equiv 2 \pmod 4, \\ (2^{2f_2} - 1)^{(p-1)/2f_2} & \textit{if } f_2 \equiv 1 \pmod 2. \end{cases}$$

We conclude this paper with quoting another formula for $h^-_{p,-3}$ expressed as a product of two determinants of degree r which is to be compared with the above one and is proved from the results of [3], [4] by the same way as used in [5]. For a rational integer a prime to p, we denote by $R^{(3)}(a)$ a residue of a modulo p such that $-2p/3 < R^{(3)}(a) < 2p/3$ and $R^{(3)}(a) \equiv 0 \pmod 2$ or $-p/3 < R^{(3)}(a) < p/3$, and put

$$D_p^{(3)} = |\, R^{(3)}(ab')\,|_{1 \leq a,b \leq r}.$$

Further let

$$V(a) = \begin{cases} 1 & \text{if } R^{(3)}(a) \equiv 0 \pmod 2, \\ -2 & \text{if } R^{(3)}(a) \equiv 1 \pmod 2 \end{cases}$$

and put

$$V_p = |\, V(ab')\,|_{1 \leq a,b \leq r}.$$

Then the following holds:

$$|\, D_p^{(3)} \,| = \begin{cases} 2^{(p-1)/f_3} p^r \,|\, \prod_{i=1}^{r} \frac{1}{2} B_{1,\psi^{2i-1}} \,| & \\ & \text{if } f_3 \equiv 0 \pmod 2, \\ 0 & \text{if } f_3 \equiv 1 \pmod 2, \end{cases}$$

$$|\, V_p \,| = \begin{cases} 2 \cdot 3^r \,|\, \prod_{i=1}^{r} \frac{1}{2} B_{1,\psi^{2i}\chi} \,| & \text{if } p \equiv 2 \pmod 3, \\ 0 & \text{if } p \equiv 1 \pmod 3, \end{cases}$$

$$\frac{1}{(3p)^{r-1}} |\, D_p^{(3)} \cdot V_p \,| = \begin{cases} 2^{(p-1)/f_3 - 1} h^-_{p,-3} & \\ & \text{if } p \equiv 5 \pmod{12}, \\ 0 & \text{if } p \not\equiv 5 \pmod{12}, \end{cases}$$

where $\chi \in X - \{\chi_0\}$.

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