On Subsets of C^{n+1} in General Position

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(Communicated by Kiyosi ITÔ, M. J. A., March 12, 1996)

1. Introduction. Let X be a subset of C^{n+1} such that $\# X \geq n+1$. It is said that X is in general position if any $n+1$ elements of X are linearly independent. From now on throughout the paper we suppose that X is in general position. Put

$$X(0) = \{\boldsymbol{a} = (a_1, \ldots, a_n, a_{n+1}) \in X : a_{n+1} = 0\} \text{ and } \nu = \# X(0).$$

It is easily seen that $0 \leq \nu \leq n$. We introduce the following notion to refine the fundamental inequality of H. Cartan for holomorphic curves ([1]).

Definition 1. We say that

(i) X is maximal (in the sense of general position) if and only if for any Y in general position such that $X \subset Y \subset C^{n+1}$, $X = Y$.

(ii) X is ν-maximal if X is maximal and $\# X(0) = \nu$.

The purpose of this paper is to give an example of ν-maximal subset of C^{n+1} for any $\nu (1 \leq \nu \leq n)$. Applications to the value distribution theory of holomorphic curves ([2], [3]) will appear elsewhere.

2. Lemma. We use the following notation.

(a) The difference product of $x_1, \ldots, x_n$:

$$\Delta_n = \Delta(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j) = \begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{vmatrix}.$$

(b) The elementary symmetric polynomials in $x_1, \ldots, x_n$:

$$\sigma_{n0} = \sigma_0(x_1, \ldots, x_n) = 1,$$
$$\sigma_{n1} = \sigma_1(x_1, \ldots, x_n) = x_1 + \cdots + x_n,$$
$$\sigma_{n2} = \sigma_2(x_1, \ldots, x_n) = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n,$$
$$\cdots\cdots$$
$$\sigma_{nn-1} = \sigma_{n-1}(x_1, \ldots, x_n) = x_1 \cdots x_{n-1} + x_1 \cdots x_{n-2}x_n + \cdots + x_2 \cdots x_n,$$
$$\sigma_{nn} = \sigma_n(x_1, \ldots, x_n) = x_1 \cdots x_n,$$
$$\sigma_{nn+1} = \sigma_{n+1}(x_1, \ldots, x_n) = 0.$$

(c) We put

$$f_{nj}(t) = \sigma_j(1, t, \ldots, t^{n-1}) \ (j = 0, 1, \ldots, n+1).$$

These f_{nj} are polynomials, which are not identically equal to zero except f_{nn+1}.

Lemma 1. For $j = 1, \ldots, n$,

$$\sigma_j(x_1, \ldots, x_n) = \sigma_j(x_1, \ldots, x_{n-1}) + \sigma_{j-1}(x_1, \ldots, x_{n-1})x_n.$$

This is easily seen from the definition of the elementary symmetric polynomials.

Lemma 2.

$$(1) \quad \begin{vmatrix} \alpha_1 & \alpha_2 & \cdots\cdots & \alpha_n & \alpha_{n+1} \\ x_1^n & x_1^{n-1} & \cdots\cdots & x_1 & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ x_n^n & x_n^{n-1} & \cdots\cdots & x_n & 1 \end{vmatrix}$$
$$= \Delta_n \sum_{j=1}^{n+1} (-1)^{j-1} \sigma_{nj-1} \alpha_j$$
$$= \Delta_n \sum_{j=1}^{n} (-1)^{j-1} \sigma_{n-1 j-1} \cdot (\alpha_j - x_n \alpha_{j+1})$$

The first equality is well-known and we can prove the second one by Lemma 1.

Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1}$ be the standard basis of C^{n+1}.

Lemma 3. For any vector $(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ of C^{n+1} which is not equal to 0, $\alpha \boldsymbol{e}_1$, or $\beta \boldsymbol{e}_{n+1}$, there exist complex numbers $a_1, \ldots, a_n$ different from each other for which the vectors

$$(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}), \ (a_1^n, \ldots, a_1, 1), \ldots, (a_n^n, \ldots, a_n, 1)$$

are linearly dependent, where α and β are any complex numbers.

Proof. We have only to find $x_j = a_j (j = 1, \ldots, n)$ different from each other for which the determinant (1) reduces to zero.

(a) The case when at least two of $\alpha_1, \ldots, \alpha_{n+1}$ are different from zero.

Put $x_k = t^{k-1}x_1$ $(k = 2, \ldots, n)$ and substitute them into (1). Then

$$\sigma_j(x_1, \ldots, x_n) = x_1^j f_{nj}(t) \ (j = 1, \ldots, n)$$

and the right-hand side of (1) divided by Δ_n is equal to

$$(2) \quad \alpha_1 - \alpha_2 f_{n1}(t)x_1 + \cdots + (-1)^{n-1}\alpha_n f_{nn-1}(t)x_1^{n-1} + (-1)^n \alpha_{n+1} f_{nn}(t) x_1^n.$$

Let $t = t_0 (\neq 0)$ be any number for which

$t_0^k \neq 1$ $(k = 1, \ldots, n-1)$ and
$f_{nj}(t_0) \neq 0$ $(j = 1, \ldots, n)$
and substitute it into (2). Then, (2) reduces to a polynomial in x_1 (say, $g(x_1)$) of degree $l (1 \leq l \leq n)$ with at least two terms. We consider the equation $g(x) = 0$. This equation has at least one non-zero solution by the choice of t_0 and since $g(x)$ has at least two terms. Let a_1 be one of its non-zero solutions and put

$$a_k = t_0^{k-1} a_1 \quad (k = 2, \ldots, n).$$

Then, $a_1, \ldots, a_n$ are different from each other and the determinant (1) reduces to zero for $x_j = a_j (j = 1, \ldots, n)$.

(b) The case when only one of $\alpha_2, \ldots, \alpha_n$ is different from zero.

Let $\alpha_j \neq 0$ $(2 \leq j \leq n)$. Then, the determinant (1) is equal to

(3) $(-1)^{j-1} \Delta(x_1, \ldots, x_n) \sigma_{j-1}(x_1, \ldots, x_n) \alpha_j$.

Substitute $x_k = a_k = k$ $(k = 1, \ldots, n-1)$ into (3) to obtain

$\sigma_{j-1}(x_1, \ldots, x_n) = \sigma_{j-1}(1, 2, \ldots, n-1, x_n) = p + qx_n$,

where p and q are positive integers. We take $x_n = a_n = -p/q$. Then, $a_1, \ldots, a_n$ are different from each other and the determinant (1) reduces to zero for these $x_j = a_j$ $(j = 1, \ldots, n)$.

3. Result.

Let

$V_n = \{(a^n, a^{n-1}, \ldots, a, 1) : a \in \boldsymbol{C}\} \cup \{\boldsymbol{e}_1\}$.

Proposition 1. V_n is 1-maximal.

Proof. It is easy to see that V_n is in general position and $\# V_n(0) = 1$. We have only to prove that for any vector

$$\boldsymbol{x} = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \ (\neq \boldsymbol{0})$$

which does not belong to V_n, $V_n \cup \{\boldsymbol{x}\}$ is not in general position.

(a) When $\alpha_1 \neq 0$, $\alpha_2 = \cdots = \alpha_{n+1} = 0$ or $\alpha_1 = \cdots = \alpha_n = 0$, $\alpha_{n+1} \neq 0$, it is trivial that $V_n \cup \{\boldsymbol{x}\}$ is not in general position.

(b) When $\boldsymbol{x} \neq \alpha \boldsymbol{e}_1$ or $\beta \boldsymbol{e}_{n+1}$ for any complex numbers α and β, $V_n \cup \{\boldsymbol{x}\}$ is not in general position by Lemma 3.

Proposition 2. For any p $(1 \leq p \leq n-1)$, the set

$U_{n,p} = \{(a^n, a^{n-1}, \ldots, a^2, a, a^p + 1) : a \in \boldsymbol{C}\} \cup \{\boldsymbol{e}_1\}$

is $p + 1$-maximal.

Proof. It is easy to see that

(a) $\# U_{n,p}(0) = p + 1$ and (b) $U_{n,p}$ is in general position.

We have only to prove that for any vector

$$\boldsymbol{x} = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \ (\neq \boldsymbol{0})$$

which does not belong to $U_{n,p}$, the set $U_{n,p} \cup \{\boldsymbol{x}\}$ is not in general position. In fact the vector

$$\boldsymbol{x}' = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1} - \alpha_{n+1-p})$$

does not belong to V_n and $V_n \cup \{\boldsymbol{x}'\}$ is not in general position by Proposition 1 and so $U_{n,p} \cup \{\boldsymbol{x}\}$ is not in general position.

Propositions 1 and 2 give us the following

Theorem 1. For any ν $(1 \leq \nu \leq n)$, there is a ν-maximal subset of $\boldsymbol{C}^{n+1}$ in the sense of general position.

Remark. There is no 0-maximal subset of $\boldsymbol{C}^2$ in the sense of general position.

In fact, suppose that there is a 0-maximal subset X of $\boldsymbol{C}^2$ in the sense of general position. Then, for any vector $\boldsymbol{x} = (\alpha, \beta)$ of X such that $\beta \neq 0$, $\boldsymbol{e}_1$ and $\boldsymbol{x}$ are then linearly independent. This shows that X is not 0-maximal, which is a contradiction.

Problem. Is there any 0-maximal subset of $\boldsymbol{C}^{n+1}$ in the sense of general position for $n \geq 2$?

4. Extension.

The purpose of this section is to extend the result obtained for subsets of $\boldsymbol{C}^{n+1}$ in Section 3 to sets of holomorphic curves.

Let Γ be a field of meromorphic functions in the complex plane and h a holomorphic curve from $\boldsymbol{C}$ into $P^n(\boldsymbol{C})$ with a reduced representation

$$(h_1, \ldots, h_{n+1}) : \boldsymbol{C} \to \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}.$$

Further, we put

$\boldsymbol{H}(\Gamma) = \{h = [h_1, \ldots, h_{n+1}] : h_j / h_k \in \Gamma (j = 1, \ldots, n+1)$ for some k $(1 \leq k \leq n+1)\}$.

Then, $\boldsymbol{H}(\Gamma) \supset P^n(\boldsymbol{C})$. Let X be a subset of $\boldsymbol{H}(\Gamma)$ such that $\# X \geq n+1$. It is said that X is in general position if for any $n+1$ elements $h_q = [h_{1q}, \ldots, h_{n+1q}]$ $(q = 1, \ldots, n+1)$ of X, $\det(h_{jq}) \neq 0$.

From now on throughout the section we suppose that X is in general position. We set

$X(0) = \{h = [h_1, \ldots, h_n, h_{n+1}] \in X : h_{n+1} = 0\}$ and $\# X(0) = \nu$.

It is easily seen that $0 \leq \nu \leq n$ since X is in general position.

As in the case of $\boldsymbol{C}^{n+1}$, we give a definition for X.

Definition 2. We say that

(i) X is maximal (in the sense of general position) if and only if for any Y in general position such that $X \subset Y \subset \boldsymbol{H}(\Gamma)$, $X = Y$.

(ii) X is ν-maximal if X is maximal and $\# X(0) = \nu$.

We shall give an example of ν-maximal subset of $\boldsymbol{H}(\Gamma)$ for any $\nu(1 \leq \nu \leq n)$. We set $\Gamma^{n+1} = \{(\alpha_1,\ldots, \alpha_{n+1}) : \alpha_j \in \Gamma\,(j=1,\ldots, n+1)\}$.

Lemma 4. For any $(\alpha_1, \ldots, \alpha_{n+1}) \in \Gamma^{n+1}$ which is not equal to $\mathbf{0}$, $\alpha \boldsymbol{e}_1$ or $\beta \boldsymbol{e}_{n+1}$, where α and β are any elements of Γ, there exist $a_1, \ldots, a_n$ in Γ different from each other for which the determinant of vectors

$$(\alpha_1,\ldots, \alpha_{n+1}),\ (a_1^n,\ldots, a_1, 1), \ldots, (a_n^n, \ldots, a_n, 1)$$

is identically equal to zero.

Proof. We shall find $x_j = a_j \in \Gamma\,(j=1, \ldots, n)$ for which the determinant (1) reduces to zero identically.

(a) The case when at least two of $\alpha_1, \ldots, \alpha_{n+1}$ are not identically equal to zero.

We first determine $x_1 = a_1$ as follows. Put

$$x_n = t^l x_1\ (0 \leq l \leq n-2) \text{ and } x_k = t^{k-1} x_1 \quad (k=2,\ldots, n-1),$$

and substitute them into (1). Then by Lemma 1 the determinant (1) devided by Δ_n is equal to

$$(4)\quad \alpha_1 + \sum_{j=2}^{n+1} (-1)^{j-1}\{t^l f_{n-1j-2}(t) + f_{n-1j-1}(t)\}\alpha_j x_1^{j-1}$$

$$(5)\quad = \sum_{j=1}^{n} (-1)^{j-1} f_{n-1j-1}(t)\alpha_j x_1^{j-1} - t^l \sum_{j=1}^{n} (-1)^{j-1} f_{n-1j-1}(t)\alpha_{j+1} x_1^{j}.$$

Let $t = t_0 (\neq 0)$ be any number for which

$$t_0^k \neq 1\ (k=1,\ldots, n-2),\ f_{n-1j-1}(t_0) \neq 0$$

$(j=2, \ldots, n)$ and $t_0^l f_{n-1j-2}(t_0) + f_{n-1j-1}(t_0) \neq 0$ $(l=0,\ldots, n-2\,; j=2,\ldots, n+1)$.

Substitute $t = t_0$ into (4) and (5), then we have

$$(6)\quad G_l(x_1) = \alpha_1 + \sum_{j=2}^{n+1} (-1)^{j-1}\{t_0^l f_{n-1j-2}(t_0) + f_{n-1j-1}(t_0)\}\alpha_j x_1^{j-1} = G(x_1) - t_0^l x_1 H(x_1),$$

where

$$G(x_1) = \sum_{j=1}^{n} (-1)^{j-1} f_{n-1j-1}(t_0)\, \alpha_j x_1^{j-1},$$

$$H(x_1) = \sum_{j=1}^{n} (-1)^{j-1} f_{n-1j-1}(t_0)\, \alpha_{j+1} x_1^{j-1}.$$

There are only a finite number of functions in Γ which satisfy

(7) $G_l(x_1) = 0\ (l=0,1,\ldots, n-2)$ and $H(x_1) = 0$,

so that we can find $x_1 = a_1$ in Γ which does not satisfy any equations in (7):

(8) $G_l(a_1) \neq 0\ (l=0,\ldots, n-2)$ and $H(a_1) \neq 0$.

Next, We choose $x_k = a_k\ (k=2,\ldots, n-1)$ and $x_n = a_n$ as follows:

$$(9)\quad x_k = a_k = t_0^{k-1} a_1 \quad (k=2,\ldots, n-1) \text{ and } x_n = a_n = G(a_1)/H(a_1).$$

Then, by the choice of t_0 and $x_1 = a_1$, $a_1, \ldots, a_n$ are different from each other and they belong to Γ. In fact,

(i) It is easy to see that $a_1,\ldots, a_{n-1}$ are different from each other.

(ii) $a_n \neq a_k\ (k=1,\ldots, n-1)$. Contrary to the assertion, if $a_n = a_k$ for some $k\ (1 \leq k \leq n-1)$, then $a_n = t_0^l a_1\ (l = k-1)$ and from (6) and (9)

$$G_l(a_1) = G(a_1) - t_0^l a_1 H(a_1) = G(a_1) - a_n H(a_1) = 0,$$

which is contrary to (8).

(iii) It is easy to see that $a_1,\ldots, a_{n-1}$ belong to Γ since a_1 does.

(iv) a_n belongs to Γ since $\alpha_1, \ldots, \alpha_{n+1}, a_1 \in \Gamma$ and Γ is a field.

For these $x_j = a_j (j=1, \ldots, n)$, the determinant (1) is equal to

$$\Delta(a_1,\ldots, a_n)\{G(a_1) - a_n H(a_1)\}$$

as in (6), which is identically equal to 0 by (9).

(b) The case when only one of $\alpha_2, \ldots, \alpha_n$ is not identically equal to zero.

Let $\alpha_j\ (2 \leq j \leq n)$ be not identically equal to zero. Then, as in the case of Lemma 3, Case (b), for

$$x_k = a_k = k \quad (k=1,\ldots, n-1) \text{ and } x_n = a_n = -u/v,$$

where $\sigma_{j-1}(1, \ldots, n-1, x_n) = u + v x_n$, u and v being positive integers, the determinant (1) is identically equal to zero.

Let $V_n(\boldsymbol{C}) = \{[a^n, \ldots, a, 1] : a \in \boldsymbol{C}\} \cup \{[1,0,\ldots, 0]\}$ and

$\boldsymbol{H}_n(\Gamma) = \{[f^n, f^{n-1}g, \ldots, fg^{n-1}, g^n] : f$ and g are entire functions without common zeros such that $f/g \in \Gamma (g \neq 0)$ or $f = 1\ (g=0)\}$.

Then, $\boldsymbol{H}_n(\Gamma) \supset V_n(\boldsymbol{C})$.

Proposition 3. $\boldsymbol{H}_n(\Gamma)$ is 1-maximal.

Proof. It is clear that $\boldsymbol{H}_n(\Gamma)$ is in general position and $\#\boldsymbol{H}_n(\Gamma)(0) = 1$. We have only to prove that for any holomorphic curve

$$h = [h_1,\ldots, h_{n+1}] \in \boldsymbol{H}(\Gamma)$$

which does not belong to $\boldsymbol{H}_n(\Gamma)$, $\boldsymbol{H}_n(\Gamma) \cup \{h\}$ is not in general position.

(a) When $h_1 \neq 0$, $h_2 = \cdots = h_{n+1} = 0$ or $h_1 = 0,\ldots, h_n = 0$, $h_{n+1} \neq 0$, it is easy to see that the set is not in general position.

(b) The other cases. The set is not in general position by Lemma 4 since

$$(f^n, f^{n-1}g,\ldots, fg^{n-1}, g^n)/g^n \in \Gamma^{n+1} \text{ and } (h_1,\ldots, h_{n+1})/h_k \in \Gamma^{n+1}$$

for $g \neq 0$ and $h_k \neq 0$.

Proposition 4. For any $p\ (1 \leq p \leq n-1)$, the set

$$\boldsymbol{H}_{np}(\Gamma) = \{[f^n, f^{n-1}g, \dots, fg^{n-1}, g^n + f^p g^{n-p}] : [f^n, \dots, fg^{n-1}, g^n] \in \boldsymbol{H}_n(\Gamma)\}$$

is $p + 1$-maximal.

Proof. It is easy to see that

(a) $\# \boldsymbol{H}_{np}(\Gamma)(0) = p + 1$ and

(b) $\boldsymbol{H}_{np}(\Gamma)$ is in general position.

We have only to prove that for any holomorphic curve

$$h = [h_1, \dots, h_{n+1}] \in \boldsymbol{H}(\Gamma)$$

which does not belong to $\boldsymbol{H}_{np}(\Gamma)$, the set $\boldsymbol{H}_{np}(\Gamma) \cup \{h\}$ is not in general position. In fact, the holomorphic curve

$$h' = [h_1, \dots, h_n, h_{n+1} - h_{n+1-p}] \in \boldsymbol{H}(\Gamma)$$

does not belong to $\boldsymbol{H}_n(\Gamma)$ and $\boldsymbol{H}_n(\Gamma) \cup \{h'\}$ is not in general position by Proposition 3 and so, $\boldsymbol{H}_{np}(\Gamma) \cup \{h\}$ is not in general position.

Propositions 3 and 4 give us the following

Theorem 2. For any ν $(1 \le \nu \le n)$, there is a ν-maximal subset of $\boldsymbol{H}(\Gamma)$ in the sense of general position.

References

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