Property C with Constraints for PDE

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(Communicated by Kiyosi ITÔ, M. J. A., Dec. 12, 1996)

Abstract: Let L_j, $j = 1,2$, be a pair of linear partial differential expressions in $\boldsymbol{R}^n$, $n \geq 3$, $D \subset \boldsymbol{R}^n$ be a bounded domain, $N_j := \{w : L_j w = 0 \text{ in } D\}$, N_{j,m_j} is a linear subspace in N_j of finite codimension $m_j < \infty$. We say that the pair $\{L_1, L_2\}$ has property C if the set of products $\{w_1 w_2\}$ is complete (total) in $L^p(D)$ for some $p \geq 1$. Here $w_j \in N_j$ run through subsets of N_j such that the products $w_1 w_2$ are well defined. We say that the pair $\{L_1, L_2\}$ has property C with constraints if the set $\{w_1 w_2\}$, where $w_j \in N_{j,m_j}$, $j = 1,2$, is total in $L^p(D)$. It is proved that if L_1 and L_2 have constant coefficients and the pair $\{L_1, L_2\}$ has property C then it has property C with constraints.

Key words: Property C with constraints; inverse problems; completeness of the set of products.

1. Introduction. The author introduced property C for pairs $\{L_1, L_2\}$ of linear partial differential expressions in [1] and has found many applications of this property [2]. In [3] he introduced property C with constraints and found several applications of this concept to inverse spectral problem, inverse boundary problem and inverse scattering problem.

In [2] necessary and sufficient conditions for property C to hold for a pair of linear partial differential expressions (formal differential operators) with constant coefficients are found.

The basic result of this paper is the following theorem.

Theorem 1.1. *If $\{L_1, L_2\}$ are linear formal partial differential operators in $\boldsymbol{R}^n$, $n \geq 3$, with constant coefficients and property C holds for the pair $\{L_1, L_2\}$, then property C with constraints holds for this pair.*

In section 2 we define property C and property C with constraints and recall some results from [2].

In section 3 we prove Theorem 1.1.

2. Basic definitions and known results.

2.1. Let $L_m u(x) = \sum_{|j| \leq J_m} a_{jm}(x) \partial^j u(x)$, $m = 1, 2, x \in \boldsymbol{R}^n$, $n \geq 2$, j is a multi-index, $a_{jm}(x)$ are given functions, $J_m > 0$ is an integer, $\partial^j u := \dfrac{\partial^j u}{\partial x_1^{j_1} \cdots \partial x_n^{j_n}}$, $j_1 + \cdots + j_n := |j|$. We call L_m a linear formal differential operator.

1991 Mathematics Subject Classification. 35E20, 35R30, 35P10.

Let $D \subset \boldsymbol{R}^n$ be a bounded domain, $N_m := \{w : L_m w = 0 \quad \text{in} \quad D\}$, $f \in L^p(D)$, $p \geq 1$. The equation $L_m w = 0$ is understood in distributional sense. Assume that

$$(2.1) \qquad \int_D f w_1 w_2 dx = 0, \quad w_m \in N_m$$

for all $w_m \in N_m$ for which $w_1 w_2 \in L^{p'}(D)$, $p' = \dfrac{p}{p-1}$.

Definition 2.1. *If* (2.1) *implies that $f = 0$, then we say that the pair $\{L_1, L_2\}$ has property C.*

Remark 2.1. *The name "property C" comes from "completeness of the set of products $\{w_1 w_2\}$* [2].

We give now a necessary and sufficient condition for a pair $\{L_1, L_2\}$ of operators with constant coefficients, $a_{jm}(x) = a_{jm} = const$, to have property C.

Define

$$(2.2) \qquad \mathscr{L}_m := \{z : z \in \boldsymbol{C}^n, L_m(z) = 0\}, \quad L_m(z) = \sum_{|j| \leq J_m} a_{jm} z^j.$$

Let $T_m(z_0)$ be the tangent space in $\boldsymbol{C}^n$ to the algebraic variety $\mathscr{L}_m$ at the point z_0.

Theorem 2.1. ([2,p.44]). *For a pair $\{L_1, L_2\}$ to have property C it is necessary and sufficient that there exist two points $z_m \in L_m$, such that the tangent spaces $T_m(z_m)$, $m = 1,2$, are transversal.*

Remark 2.2. *Geometrically this means that the variety $\mathscr{L}_1 \cup \mathscr{L}_2$ is not a union of parallel hyperplanes in $\boldsymbol{C}^n$.*

2.2. We now define *property C with con-*

straints. Let $N_{m,M(m)}$ be a linear subspace in N_m of finite condimension $M(m) < \infty$, $m = 1,2$. Assume that (2.1) holds for all $w_m \in N_{m,M(m)}$ such that $w_1 w_2 \in L^{p'}(D)$.

Definition 2.2. *If, under the above assumption, equation* (2.1) *implies* $f = 0$, *then we say that the pair* $\{L_1, L_2\}$ *has property* C *with constraints.*

In what follows we assume that L_m have constant coefficients. Let $z \cdot x := \sum_{j=1}^n z_j x_j$.

Remark 2.3. *The function* $\exp(z \cdot x) \in N_m$ *iff* $z \in \mathcal{L}_m$.

3. Proof of Theorem 1.1. Assume that the pair $\{L_1, L_2\}$ has property C, that is:

$$(3.1)\ \left\{0 = \int_D f(x) w_1 w_2 dx \quad \forall\, w_m \in N_m\right\} \Rightarrow f = 0.$$

We want to prove that (3.1) implies that the pair $\{L_1, L_2\}$ has property C with constraints, that is:

$$(3.2)\quad \left\{0 = \int_D f(x) w_1 w_2 dx \quad \forall\, w_m \in N_{m,M(m)}\right\} \Rightarrow f = 0.$$

Take $w_m = \exp(z_m \cdot x)$, $z_m \in \mathcal{L}_m$, $m = 1,2$. Let $g_m(z_m)$ be a smooth function on $\mathcal{L}_m$, decaying faster than $\exp(c\,|\,z_m\,|)$ for any $c > 0$, and $\sigma_m(z_m)$ be a finite measure on $\mathcal{L}_m$. The function

$$(3.3)\quad w_m(x) := \int_{\mathcal{L}_m} d\sigma_m(z_m) \exp(z_m \cdot x) g_m(z_m)$$

belongs to $N_{m,M(m)}$ provided that

$$(3.4)\quad 0 = \int_{\mathcal{L}_m} d\sigma_m g_m(z_m) H_{m\kappa}(z_m),\ 1 \le \kappa \le M(m);$$

$$H_{m\kappa}(z_m) := \langle \exp(z_m \cdot x),\, h_{m\kappa} \rangle,\ m = 1,2.$$

Here we took into account the constraints: $w_m \in N_{m,M(m)}$ implies $\langle w_m, h_{m\kappa} \rangle = 0$, $1 \le \kappa \le M(m)$, where $\langle w_m, h_{m\kappa} \rangle$ is a linear functional on N_m. If w_m are defined in (3.3), then quation (3.2) becomes

$$(3.5)\quad 0 = \int_{\mathcal{L}_1} d\sigma_1(z_1) g_1(z_1) \int_{\mathcal{L}_2} d\sigma_2(z_2) g_2(z_2)$$
$$F(z_1 + z_2);\ F(x) := \int_D dx f(x) \exp(z \cdot x),$$

where g_m satisfy (3.4).

We want to derive from (3.5) that $F(z) = 0$. This would imply $f(x) = 0$. It follows from (3.5) and (3.4) that

$$(3.6)\quad \int_{\mathcal{L}_2} d\sigma_2 g_2(z_2) F(z_1 + z_2) = \sum_{\kappa=1}^{M(1)} c_\kappa H_{1\kappa}(z_1),$$

where c_κ are some constants, $g_2(z_2)$ satisfies (3.4) with $m = 2$, and otherwise g_2 is arbitrary.

Therefore, one can choose $J > M(1)$ linearly independent functions $\varphi_j(z_2)$ and some numbers d_j, $1 \le j \le J$, such that

$$(3.7)\quad \int_{\mathcal{L}_2} g_2(z_2) H_{2\kappa}(z_2) d\sigma_2(z_2) = 0,\ 1 \le \kappa \le M(2),$$

where

$$(3.8)\qquad g_2(z_2) = \sum_{j=1}^{J} d_j \varphi_j(z_2).$$

We *claim* that, for any integer J, one can choose $\varphi_j(z_2)$ such that the functions

$$(3.9)\quad \int_{\mathcal{L}_2} d\sigma_2(z_2) \varphi_j(z_2) F(z_1 + z_2) := \omega_j(z_1),\quad 1 \le j \le J,$$

are linearly independent. If this is done, then (3.6) leads to a contradiction, unless $F(z) = 0$, or, which is the same, unless $f(x) = 0$. Indeed, the left-hand side of (3.6) is a linear span of $J > M(1)$ linearly independent functions by the *claim*, while the right-hand side of (3.6) is obviously a linear span of $M(1) < J$ linearly independent functions. These linear spaces are identical by (3.6), which contradicts the fact that they have different dimensions. This contradiction proves Theorem 1.1. To complete the proof, let us verify the above *claim*. Let $J > 0$ be an arbitrary integer, $f(x) \in L^p(D)$, $p \ge 1$, $D \subset \mathbf{R}^n$ is a bounded domain, $F(z) := \int_D f(x) \exp(z \cdot x) dx$,

$\Omega_\kappa(z_1) := \int_{\mathcal{L}_2} d\sigma_2 \omega_\kappa(z_2) F(z_1 + z_2)$, $z_m \in \mathcal{L}_m$, $m = 1,2$.

Lemma 3.1. *If* $\{L_1, L_2\}$ *has property* C *and* $f \not\equiv 0$, *then there exist* J *functions* $\omega_\kappa(z_2)$ *such that the functions* $\{\Omega_\kappa(z_1)\}_{1 \le \kappa \le J}$ *are linearly independent.*

Proof. We want to prove that $\dim R(T) = \infty$, where $\Omega(z_1) := T\omega := \int_{\mathcal{L}_2} F(z_1 + z_2) \omega(z_2) d\sigma_2(z_2)$ and $R(T) =$ range of T.

Take $\omega_j = \delta(z_2 - z_2^{(j)})$, $1 \le j \le J$, $z_2^{(j)} \in \mathcal{L}_2$, J is an arbitrary large fixed integer, and $\delta(z_2 - z_2^{(j)})$ is the delta-function. Then

$$\Omega_j := \int_D dx f(x) \exp(z_1 \cdot x + z_2^{(j)} \cdot x).$$

Let $z_m^0 \in \mathcal{L}_m$ be such that the tangent spaces T_m to $\mathcal{L}_m$ at z_m^0 are transversal, $m = 1,2$. Let $\{e_1, \cdots, e_{n-1}\}$ be an orthonormal basis in T_1 and, since by the assumption T_1 is transversal to T_2, there is a vector f_1 in the basis of T_2, such that $(f_1, e_n) \neq 0$. Without loss of generality (and for simplicity) one can assume that $f_1 = e_n$, $(e_\kappa, e_j) = \delta_{\kappa j}$, $1 \le j, \kappa \le n$ (the inner product is taken in $\mathbf{C}^n$ but the vectors e_κ are real-valued: they form a basis

of $\boldsymbol{R}^n$). Fix an arbitrary number ζ_n and find $\zeta_\kappa^{(j)}$, $1 \le \kappa \le n-1$, $1 \le j \le J$, such that
$z_2^{(j)} = \sum_{\kappa=1}^{n-1} \zeta_\kappa^{(j)} e_\kappa + \zeta_n e_n \in \mathscr{L}_2$, $\zeta_\kappa^{(j)} \ne \zeta_\kappa^{(j')}$ for $j \ne j'$.
Index κ denotes the coordinate component and j denotes the number of the chosen point.

One can choose ζ_n independently of j: the variety $\mathscr{L}_2$ is defined by the equation $L_2(\zeta) = 0$ and $\frac{\partial L_2}{\partial \zeta_n} \not\equiv 0$ on $\mathscr{L}_2$ since T_2 and T_1 are transversal. In a neighborhood of the points where $\frac{\partial L_2}{\partial \zeta_n} \ne 0$ one can write the equation of $\mathscr{L}_2$ as $\zeta_n = \phi(\zeta')$, $\zeta' := (\zeta_1, \cdots, \zeta_{n-1})$. Thus, for a fixed ζ_n, one can find, in general, infinitely many points ζ' such that $(\zeta', \zeta_n) \in \mathscr{L}_2$, provided that $n \ge 3$. Fix J such points.

Suppose that ζ_n is so chosen that the function $h(x', \zeta_n) := \int_a^b f(x', x_n) \exp(i\zeta_n x_n) dx_n \not\equiv 0$. Here $x' := (x_1, \dots, x_{n-1})$, and $[a, b]$ is a finite interval since D is a bounded domain. Clearly there exists an open set of the numbers ζ_n such that $h(x', \zeta_n) \not\equiv 0$. Indeed, $h(x', \zeta_n)$ is an entire function of ζ_n which cannot vanish on open sets of the ζ_n-axis unless $f(x', x_n) \equiv 0$, and we assumed that $f(x) \not\equiv 0$. With the above choice of $\zeta^{(j)}$, $1 \le j \le J$, and ζ_n, one has

(3.10)
$$\Omega_j = \int_D dx f(x', x_n) \exp(\zeta_n x_n) \exp[(z_1 + \zeta^{(j)}) \cdot x'] = \int_{D'} dx' h(x', \zeta_n) \exp[(z_1 + \zeta^{(j)}) \cdot x']$$
where $z_1 \in \mathscr{L}_1$ is arbitrary, $\zeta^{(j)} + \zeta_n e_n \in \mathscr{L}_2$, and D' is the (parallel to e_n) projection of D onto $\boldsymbol{R}^{n-1}$. Without loss of generality one may assume that D is a cylinder $D' \times [a, b]$. In a neighborhood $\mathscr{N}(z_j^0)$ of z_1^0 the points $z_1 \in \mathscr{L}_1$ are very close to T_1. The function Ω_j is an entire function of the variable z_1. This variable runs through an open set in $\boldsymbol{C}^{n-1}$ when z_1 runs through $\mathscr{N}(z_1^0)$. Thus, if

$$0 = \sum_{j=1}^J c_j \Omega_j := \int_{D'} dx' h(x', \zeta_n) \sum_{j=1}^J c_j \exp((\zeta^{(j)}) \cdot x') \exp(z_1 \cdot x'), \tag{3.11}$$
$\forall z_1 \in \mathscr{N}(z_1^0)$, then
$$h(x', \zeta_n) \sum_{j=1}^J c_j \exp(\zeta^{(j)}) \cdot x') \equiv 0 \quad \forall x'. \tag{3.12}$$
Since $h(x', \zeta_n) \not\equiv 0$, it follows that
$$\sum_{j=1}^J c_j \exp(\zeta^{(j)} \cdot x') = 0 \quad \forall x'. \tag{3.13}$$
Since $\zeta^{(j)} \ne \zeta^{(i)}$ for $j \ne i$, equation (3.13) implies $c_j = 0$, $1 \le j \le J$. Therefore the set $\{T\omega_j\}$, $1 \le j \le J$, is linearly independent. Since $J > 0$ is arbitrary, this means that $\dim R(T) = \infty$. Lemma 3.1 is proved. □

Therefore Theorem 1.1 is proved. □

References

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