Criterion of Wiener Type for Minimal Thinness on Covering Surfaces

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Introduction. M. Lelong [6] and L. Naïm [8] obtained a criterion of Wiener type for minimal thinness for the Martin compactification of the upper half space of the d-dimensional Euclidean space $(d > 1)$. The purpose of this note is to give a criterion of Wiener type for minimal thinness for the Martin compactification of a finite sheeted covering surface of a punctured Riemannian sphere. It is sufficient to consider an r-sheeted unlimited covering surface W of $D - \{0\}$ (D is the unit disc). Denote by ∂W the relative boundary of W and $\pi = \pi_W$ the projection of $\bar{W} = W \cup \partial W$ onto $\{0 < |z| \leq 1\}$. We consider the Martin compactification W^* of W. Then W^* takes a form $W^* = W \cup \partial W \cup \Delta$, where Δ is the *ideal boundary* of a bordered surface $\bar{W}$. We also denote by Δ_1 the set of minimal points in Δ. We note that $1 \leq \# \Delta_1 \leq r$, where $\# \Delta_1$ is the number of points in Δ_1 (cf. [4]). Let $\Delta_1 = \{\zeta_1, \ldots, \zeta_m\}$ $(m = \# \Delta_1)$ and denote by $k_j = k_{\zeta_j}$ $(j = 1, \ldots, m)$ the Martin function with pole at ζ_j. We set $U_j = \{w \in W : k_j(w) > \sum_{i \neq j} k_i(w)\}$ $(j = 1, \ldots, m)$ in the case of $m > 1$ and $U_1 = W$ in the case of $m = 1$.

Main theorem. *Let E be a subset of W and j be an integer with $1 \leq j \leq m$. Set $E_n = \{w \in E \cap U_j : s^n \leq k_j(w) \leq s^{n+1}\}$ $(s > 1)$. Then, E is minimally thin at ζ_j if and only if*

$$\sum_{n=1}^{\infty} cap_W(E_n) s^n < +\infty,$$

where $cap_W(E_n)$ is the outer Green capacity of E_n.

1. Preliminaries 1.1 We begin with recalling the definition of balayage. Consider an open Riemann surface F possessing the Green function. Denote by $\mathcal{S} = \mathcal{S}(F)$ the class of all nonnegative superharmonic functions on F. Let E be a subset of F and s belong to $\mathcal{S}$. Then the *balayage* $\hat{R}_s^E = {}^F\hat{R}_s^E$ of s relative to E on F is defined by

$$\hat{R}_s^E(z) = \liminf_{x \to z} \inf\{u(x) : u \in \mathcal{S},\ u \geq s \text{ on } E\}$$

(cf. e.g. [2]). For informations about fundamental properties of balayage we refer to [1],[2], [5], etc.

The following lemma gives us the relation between the balayage on F and that on a covering surface of F.

Lemma 1.1 (cf. [7]**).** *Let $\tilde{F}$ be an unlimited covering surface of F, E a subset of F, s a positive superharmonic function on F and π the canonical projection from $\tilde{F}$ onto F. Then, it holds that*

$${}^F\hat{R}_s^E \circ \pi = {}^{\tilde{F}}\hat{R}_{s\circ\pi}^{\pi^{-1}(E)}$$

on $\tilde{F}$.

Next we state the definition of thinness (cf. [1]). Let G_z^F be the Green function on F with pole at z.

Definition 1.1. *Let z be a point of F and E a subset of F. We say that E is thin at z if ${}^F\hat{R}_{G_z^F}^E \neq G_z^F$ on F.*

Assuming that E is closed and z belongs to E in the above definition, it is well-known that E is thin at z if and only if z is an irregular point of $F - E$ with respect to Dirichlet problem (cf. e.g. [2]). In the case of $F = D = \{z \in \mathbf{C} : |z| < 1\}$ we here review the Wiener criterion for thinness.

Proposition 1.1 (cf. [1]). *Let L be a subset of D. Set*

$$L_n = \{z \in L : s^n \leq \log|z|^{-1} \leq s^{n+1}\}\ (s > 1).$$

Then, L is thin at 0 if and only if

$$\sum_{n=1}^{\infty} cap_D(L_n) s^n < +\infty,$$

where $cap_D(L_n)$ is the outer Green capacity of L_n.

1.2. First we begin with definition of minimal thinness. Let k_ζ be the Martin function on F with pole at $\zeta \in \Delta_1^F$.

Definition 1.2 (cf. [1]). Let ζ be a point of Δ_1^F and E a subset of F. Then, we say that E is *minimally thin* at ζ if $\hat{R}_{k_\zeta}^E \neq k_\zeta$ on F.

Definition 1.3. Let ζ be a point of Δ_1^F and U a subset of F. We say that $U \cup \{\zeta\}$ is *a minimal fine neighborhood* of ζ if $F - U$ is minimally thin at ζ.

We close Preliminaries by stating the following (cf.[8], [3]).

Proposition 1.2. *Let ζ be a point of Δ_1^F and E a subset of F. Then, E is minimally thin at ζ if and only if ${}^F\hat{R}_{k_\zeta}^E$ is a Green potential on F.*

2. Proof of the main theorem 2.1 For simplicity of notation we denote by $\hat{R}_f^E$ the balayage ${}^W\hat{R}_f^E$ of $f \in \mathcal{S}$ on W and set $g_x(z) = \log\left|\dfrac{1-\bar{x}\cdot z}{z-x}\right|$, and $g = g_0$. We write by p^ν (resp. q^λ) the Green potential on $D-\{0\}$ (resp. W) of a Radon measure ν (resp. λ) on $D-\{0\}$ (resp. W). The next proposition is the heart of the main theorem.

Proposition 2.1. *Let E be a subset of W. Then, E is minimally thin at every $\zeta_j \in \Delta_1$ if and only if $\pi(E)$ is thin at 0.*

Proof. Suppose E is minimally thin at every $\zeta_j \in \Delta_1$. By Proposition 1.2 $\hat{R}_{k_j}^E (j=1,\dots,m)$ is a Green potential on W. We remark that there exist positive constants $c_j (j=1,\dots,m)$ such that $(*)$ $g\circ\pi = \sum_{j=1}^m c_j \cdot k_j$ on W. Hence, $\hat{R}_{g\circ\pi}^E = \sum_{j=1}^m c_j \cdot \hat{R}_{k_j}^E$ is a Green potential on W. Let μ the Radon measure on W with $\hat{R}_{g\circ\pi}^E = q^\mu$ and denote by $\pi(\mu)$ the image measure of μ by π. By the fact that $g_z\circ\pi = \sum_{\pi(w)=z} n(w)\cdot G_w$ ($n(w)$ is the multiplicity of π at w), we have

$$p^{\pi(\mu)}(Z) = \int g_z\circ\pi d\mu = \int \sum_{\pi(w)=z} n(w)\cdot G_w d\mu$$

$$= \sum_{\pi(w)=z} n(w)\cdot q^\mu(w) = \sum_{\pi(w)=z} n(w)\cdot \hat{R}_{g\circ\pi}^E(w)$$

on $D-\{0\}$. Hence, by the routine argument $p^{\pi(\mu)} \geq {}^{D-\{0\}}\hat{R}_g^{\pi(E)}$ on $D-\{0\}$ because the image of a polar subset of W by π is polar. Since $p^{\pi(\mu)}$ is a Green potential on $D-\{0\}$, ${}^{D-\{0\}}\hat{R}_g^{\pi(E)}$ is a Green potential on $D-\{0\}$, and hence, ${}^D\hat{R}_g^{\pi(E)} = {}^{D-\{0\}}\hat{R}_g^{\pi(E)} \neq g$ on $D-\{0\}$. Hence, $\pi(E)$ is thin at 0.

Conversely suppose that $\pi(E)$ is thin at 0. Considering $\pi(E)$ as a subset of $D-\{0\}$ we find that $\pi(E)$ is minimally thin at 0. By Proposition 1.2 ${}^{D-\{0\}}\hat{R}_g^{\pi(E)}$ is a potential on $D-\{0\}$. By this fact it is easily checked that ${}^{D-\{0\}}\hat{R}_g^{\pi(E)}\circ\pi$ is a potential on W. By Lemma 1.1 and the equation $(*)$

$${}^D\hat{R}_g^{\pi(E)}\circ\pi = {}^{D-\{0\}}\hat{R}_g^{\pi(E)}\circ\pi$$

$$= \hat{R}_{g\circ\pi}^{\pi^{-1}(\pi(E))} \geq c_j\cdot\hat{R}_{k_j}^{\pi^{-1}(\pi(E))} \geq c_j\cdot\hat{R}_{k_j}^E$$

on W. Hence, $\hat{R}_{k_j}^E$ is a potential on W and hence, E is minimally thin at every $\zeta_j \in \Delta_1$. Therefore we have the desired result.

2.2. Before proceeding to the proof of the main theorem we observe some preliminary facts.

Lemma 2.1. *Let $U_j (j=1,\dots,m)$ be as in Introduction. Then, $U_j \cup \{\zeta_j\} (j=1,\dots,m)$ is a minimal fine neighborhood of ζ_j with $U_i \cap U_j = \emptyset$ $(i\neq j)$.*

Proof. By the definition of U_j we have $U_i \cap U_j = \emptyset$ $(i\neq j)$, and

$$\hat{R}_{k_j}^{W-U_j} = \hat{R}_{k_j}^{\{k_j\leq\sum_{i\neq j}k_i\}} \leq \sum_{i\neq j} k_i < k_j$$

on U_j. Therefore we have the desired result.

By the definition of U_j and the fact that $g\circ\pi = \sum_{i=1}^m c_i\cdot k_i (c_i>0,\ i=1,\dots,m)$ on W we have

Lemma 2.2. *$k_j (j=1,\dots,m)$ is comparable with $g\circ\pi$ on U_j, that is, there exist positive constants A and B such that*

$$A\cdot k_j \leq g\circ\pi \leq B\cdot k_j \quad \text{on } U_j.$$

Lemma 2.3. *Let K be a subset of W. Then,*

$$cap_D(\pi(K)) \leq cap_W(K) \leq r\cdot cap_D(\pi(K)).$$

Proof. We may suppose that K is a compact subset of W. We remark that $\hat{R}_1^K$ (resp. ${}^{D-\{0\}}\hat{R}_1^{\pi(K)}$) is a Green potential on W (resp. $D-\{0\}$). Let μ_K (resp. $\mu_{\pi(K)}$) be the Radon measure on W (resp. $D-\{0\}$) with $\hat{R}_1^K = q^{\mu_K}$ (resp. ${}^{D-\{0\}}\hat{R}_1^{\pi(K)} = p^{\mu_{\pi(K)}}$). Since $1 \leq p^{\pi(\mu_K)}(z) = \sum_{\pi(w)=z} n(w)\cdot \hat{R}_1^K(w) \leq r$ q.e. on $\pi(K)$ ($n(w)$ is the multiplicity of π at w), and $p^{\mu_{\pi(K)}} = 1$ q.e. on $\pi(K)$, by the domination principle,

$$p^{\mu_{\pi(K)}} \leq p^{\pi(\mu_K)} \leq r\cdot p^{\mu_{\pi(K)}}$$

on D. Therefore we have the desired result (cf. [3, Corollary 4.5]).

Proof of the main theorem. By Lemma 2.1 we find that E is minimally thin at ζ_j if and only if $E\cap U_j$ is minimally thin at every ζ_i. Hence, by Proposition 2.1, E is minimally thin at ζ_j if and only if $\pi(E\cap U_j)$ is thin at 0. Therefore, by Lemmas 2.2 and 2.3, and Proposition 1.1, we have the desired result.

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