On Modality of Lévy Processes Corresponding to Mixtures of Two Exponential Distributions

By Kouji YAMAMURO

Department of Mathematics, School of Science, Nagoya University

(Communicated by Kiyosi ITÔ, M. J. A., May 12, 1995)

1. Introduction. By a Lévy process we mean a stochastically continuous stochastic process taking values in the real line $\boldsymbol{R}$, having stationary independent increments, and starting at the origin. In this paper we consider multimodality of some Lévy process. We use the following definition of strict k-modality given by Sato [5]. The restriction of a σ-finite measure μ on $\boldsymbol{R}$ to a Borel set B is denoted by $\mu|_B$.

Definition. (I) *A σ-finite measure μ on $\boldsymbol{R}$ is said to be strictly unimodal with mode a if it satisfies the following*:

(i) *The support I of μ is an interval or a singleton, and I contains a.*

(ii) *The measure $\mu|_{I\setminus\{a\}}$ has a version $f(x)$ of the density which is strictly increasing on $I \cap (-\infty, a)$ if $I \cap (-\infty, a) \neq \phi$ and strictly decreasing on $I \cap (a, \infty)$ if $I \cap (a, \infty) \neq \phi$.*

(II) *For $k \geq 2$, a σ-finite measure μ on $\boldsymbol{R}$ is said to be strictly k-modal if it satisfies the following*:

(i) *The support I of μ is an interval.*

(ii) *There are disjoint sets $I_1, \dots, I_k$ such that $I = \cup_{i=1}^{k} I_i$, each I_i is a singleton or an interval, and, for each i, $\mu|_{I_i}$ is strictly unimodal.*

(iii) *If $l < k$, then there are no disjoint sets $J_1, \dots, J_l$ such that $I = \cup_{j=1}^{l} J_j$, each J_j is a singleton or an interval, and, for each j, $\mu|_{J_j}$ is strictly unimodal.*

Strictly 2-modal is called strictly bimodal. The modes $a_1, a_2, \dots, a_k$ of $\mu|_{I_1}, \mu|_{I_2}, \dots, \mu|_{I_k}$ are called modes of μ.

Brownian motion and stable processes, which are familiar examples of one-dimensional Lévy processes, are unimodal at any time (Yamazato [9]). But there are Lévy processes which have time evolution in modality. This fact is already known to Wolfe [8] and stressed by Sato [3] [4] [5] and Watanabe [7]. Examples show that there are Lévy processes which change from unimodal to non-unimodal, or from non-unimodal to unimodal, or from unimodal to non-unimodal and again to unimodal as time passes. There are Lévy processes which change between unimodal and non-unimodal infinitely many times. Among these examples, we have few Lévy processes whose evolution in modality is completely known. One of such examples is Wolfe's (see [5] and [8]) and another is a compound Poisson process $\{X_t : t \geq 0\}$ whose distribution at $t = 1$ is

(1) $\mu = p\delta_0 + (1-p)ae^{-ax} I_{(0,\infty)}(x)\,dx,$

where δ_0 stands for the delta distribution at 0, $0 < p < 1$, $0 < a$, and $I_{(0,\infty)}(x)$ stands for the indicator function of the interval $(0, \infty)$. Sato [5] proved that the distribution of X_t is strictly unimodal for $t \leq (1+p)/(1-p)$, and strictly bimodal for $t > (1+p)/(1-p)$. The distributions of these two examples have point mass at the origin. Hence, when they are strictly unimodal, they have modes at 0 and, when they are strictly bimodal, one of their two modes is located at 0.

We would like to find out examples which do not have point mass and are strictly k-modal at some t and whose time evolution in modality can be analyzed for all time. In order to consider this problem for $k = 2$, we shall investigate modality of the Lévy process $\{X_t : t \geq 0\}$ that has the following distribution μ at $t = 1$:

(2) $\mu = (1-p)ae^{-ax} I_{(0,\infty)}(x)\,dx + pbe^{-bx} I_{(0,\infty)}(x)\,dx,$

where $0 < p < 1$ and $0 < a < b$. The distributions (1) and (2) are infinitely divisible by the result of Goldie [1], and X_t is unimodal with mode 0 for $0 < t < 1$ by the result of Steutel [6]. It is difficult to analyze modality of X_t for non-integer $t > 1$, but we can analyze it for integer $t = n$.

2. Results. From now on $\{X_t\}$ is the Lévy process that has distribution μ of (2) at $t = 1$. We shall obtain the following theorem. Denote the set of all positive integers by $\boldsymbol{N}$.

Theorem. *The distribution of X_n, $n \in \boldsymbol{N}$, is either strictly unimodal or strictly bimodal. Furth-*

ermore it is strictly unimodal if $n \geq \frac{b}{b-a}\left(1+\frac{b}{a}\frac{p}{1-p}\right)$.

Remark. Sato points out that, if $t > (1+p)/(1-p)$, then, for any $a > 0$, the distribution of X_t is non-unimodal for any sufficiently large b, because, as $b \to \infty$, X_t converges to the Lévy process that has distribution (1) at $t = 1$. Our theorem shows that, if t is an integer, then non-unimodality of the distribution of X_t implies strictly bimodal.

Before proceeding to the proof of theorem we shall state important two lemmas. In counting the number of changes of sign of a finite sequence $a_0, a_1, a_2, \ldots, a_m$, or an infinite sequence $a_0, a_1, a_2, a_3, \ldots$, we disregard zero terms (see [2], p. 36).

We can find the following lemma in [2], p. 41.

Lemma 1 (Extension of Descartes' rule of signs to power series). *Let the radius of convergence of the power series* $\sum_{l=0}^{\infty} A_l x^l$ *be* ρ. *Then the number of its zeros in* $(0, \rho)$ *does not exceed the number of changes of sign of its coefficients. Here we count the zeros according to their multiplicity.*

We can find a proposition including the following lemma in [2], p. 41.

Lemma 2. *Suppose that*

$$\sum_{l=0}^{m} a_l \frac{\lambda^l}{(\alpha-\lambda)^l} = \sum_{l=0}^{\infty} A_l \lambda^l$$

with $\alpha > 0$. *Then the number of changes of sign of* $\{A_l\}_{l\geq 0}$ *does not exceed the number of changes of sign of* $\{a_l\}_{l=0,1,\cdots,m}$.

The distribution of X_n has the following density $f_n(x)$:

$$f_n(x) = e^{-ax}\Big[(1-p)^n a^n \frac{x^{n-1}}{(n-1)!} + \sum_{l=0}^{n-2} x^l \frac{(n-2-l)!}{(b-a)^{n-l-1}} \sum_{j=l+1}^{n-1} \binom{n}{j} p^{n-j}(1-p)^j \times \frac{a^j b^{n-j}}{(j-1)!(n-j-1)!}(-1)^{j-1-l}\binom{j-1}{l}\Big]$$
$$+ e^{-bx}\Big[p^n b^n \frac{x^{n-1}}{(n-1)!} + \sum_{l=0}^{n-2} x^l \frac{(n-2-l)!}{(a-b)^{n-l-1}} \times \sum_{j=l+1}^{n-1}\binom{n}{j}(1-p)^{n-j}p^j \times \frac{b^j a^{n-j}}{(j-1)!(n-j-1)!}(-1)^{j-1-l}\binom{j-1}{l}\Big].$$

This is proved by induction. We denote by $g^{(l)}(x)$, $l \in \boldsymbol{N}$, the l-th derivative of a function $g(x)$.

Proof of Theorem. Set

$$F_n(s) = \int_0^{\infty} e^{-\frac{x}{s}} f_n(x)\,dx.$$

Then

$$F_n(s) = \left((1-p)\frac{a}{a+(1/s)} + p\frac{b}{b+(1/s)}\right)^n.$$

We have

$$F_n\left(\frac{\lambda}{b(1-\lambda)}\right) = \frac{\lambda}{b(1-\lambda)}\left(f_n(0) + \int_0^{\infty} e^{-\frac{b}{\lambda}x} e^{bx} f_n'(x)\,dx\right).$$

Here we used the form of $f_n(x)$. Set $h_n(x) = e^{bx} f_n'(x)$. In order to study modality of the distribution of X_n, we look at the number of zeros of $h_n(x)$ in $(0, \infty)$. Since $h_n(x)$ is analytic, Lemma 1 says that it is enough to look at the number of changes of sign of $\{h_n^{(l)}(0)\}_{l\geq 0}$. Now we consider the power series

$$(1-\lambda)F_n\left(\frac{\lambda}{b(1-\lambda)}\right) = \sum_{l=0}^{\infty} A_l \frac{\lambda^l}{l!}.$$

Use integration by parts repeatedly. Then,

$$\sum_{l=0}^{\infty} A_l \frac{\lambda^l}{l!} = \frac{\lambda}{b} f_n(0) + \left(\frac{\lambda}{b}\right)^2 h_n(0) + \cdots + \left(\frac{\lambda}{b}\right)^l h_n^{(l-2)}(0) + \left(\frac{\lambda}{b}\right)^l \int_0^{\infty} e^{-\frac{b}{\lambda}x} h_n^{(l-1)}(x)\,dx.$$

Differentiate both sides l times, and let $\lambda \to 0$. Then

$$A_l = \frac{l!}{b^l} h_n^{(l-2)}(0) + \lim_{\lambda\to 0}\frac{d^l}{d\lambda^l}\left(\left(\frac{\lambda}{b}\right)^l \int_0^{\infty} e^{-\frac{b}{\lambda}x} h_n^{(l-1)}(x)\,dx\right) = \frac{l!}{b^l} h_n^{(l-2)}(0).$$

Here we used the form of $h_n^{(l-1)}(x)$. Hence the number of changes of sign of $\{h_n^{(l)}(0)\}_{l\geq 0}$ is equal to the number of changes of sign of $\{A_l\}_{l\geq 2}$. On the other hand,

$$(1-\lambda)F_n\left(\frac{\lambda}{b(1-\lambda)}\right) = \lambda^n(1-\lambda)\left(\frac{(1-p)a}{b-(b-a)\lambda} + p\right)^n = p^n \lambda^n (1-\lambda)\left(\frac{\beta}{\alpha-\lambda}+1\right)^n,$$

where $\alpha = \frac{b}{b-a}$, $\beta = \frac{(1-p)a}{p(b-a)}$. Now notice that

$$(1-\lambda)\left(\frac{\beta}{\alpha-\lambda}+1\right)^n = \left(1-\frac{\lambda}{\alpha}\right)$$

$$\times \left(1-(\alpha-1)\frac{\lambda}{\alpha-\lambda}\right)\left(\frac{\beta}{\alpha}+1+\frac{\beta}{\alpha}\frac{\lambda}{\alpha-\lambda}\right)^n$$
$$=\left(1-\frac{\lambda}{\alpha}\right)\Bigg[\left(\frac{\beta}{\alpha}+1\right)^n+\sum_{l=1}^{n}\left(\frac{\beta}{\alpha}+1\right)^{n-l}\times\left(\frac{\beta}{\alpha}\right)^{l-1}\left(\frac{\lambda}{\alpha-\lambda}\right)^l\frac{n!}{(n-l+1)!l!}$$
$$\times\left\{(n-l+1)\frac{\beta}{\alpha}-(\alpha-1)l\left(\frac{\beta}{\alpha}+1\right)\right\}-(\alpha-1)\left(\frac{\beta}{\alpha}\right)^n\left(\frac{\lambda}{\alpha-\lambda}\right)^{n+1}\Bigg]$$
$$=\left(\frac{\beta}{\alpha}+1\right)^n+\lambda\Bigg[\left(\frac{\beta}{\alpha}+1\right)^{n-1}\times\frac{1}{\alpha}\left\{-\alpha\left(\frac{\beta}{\alpha}+1\right)+n\frac{\beta}{\alpha}\right\}$$
$$+\sum_{l=1}^{n-1}\left(\frac{\beta}{\alpha}+1\right)^{n-l-1}\left(\frac{\beta}{\alpha}\right)^l\frac{n!}{(n-l)!(l+1)!}\frac{1}{\alpha}\times\left\{(n-l)\frac{\beta}{\alpha}-(\alpha-1)(l+1)\left(\frac{\beta}{\alpha}+1\right)\right\}$$
$$\times\left(\frac{\lambda}{\alpha-\lambda}\right)^l-\frac{\alpha-1}{\alpha}\left(\frac{\beta}{\alpha}\right)^n\left(\frac{\lambda}{\alpha-\lambda}\right)^n\Bigg].$$

Since $\alpha > 1$ and $\beta > 0$, the coefficients of $\left(\frac{\lambda}{\alpha-\lambda}\right)^l$, $l=0, 1, \ldots, n$, in the brackets in the last expression change sign at most twice. Now we can apply Lemma 2. We see that $A_l = 0$ for $0 \le l \le n-1$, $A_n > 0$, and the sequence $\{A_l\}_{l \ge n+1}$ changes sign at most twice. Therefore, $f_n'(x)$ has at most three zeros in $(0, \infty)$. Hence X_n is either strictly unimodal or strictly bimodal.

Suppose that $n \ge \frac{b}{b-a}\left(1+\frac{b}{a}\frac{p}{1-p}\right)$, which is equivalent to

$$-\alpha\left(\frac{\beta}{\alpha}+1\right)+n\frac{\beta}{\alpha}\ge 0.$$

Then we see that $A_{n+1} \ge 0$ and that $\{A_l\}_{l \ge n+1}$ changes sign at most once. We see that, for $n \ge 2$, $f_n(0) = 0$ and $f_n'(x)$ has only one zero in $(0, \infty)$. This completes the proof of Theorem.

Corollary. *The distribution of X_n is strictly unimodal for every $n \in \mathbf{N}$ if* $\frac{a(b-2a)}{b^2} \ge \frac{p}{1-p}$.

Proof. By the latter half of Theorem, the distribution of X_n is strictly unimodal for every $n \ge 2$, if

$$2 \ge \frac{b}{b-a}\left(1+\frac{b}{a}\frac{p}{1-p}\right).$$

This condition is equivalent to $\frac{a(b-2a)}{b^2} \ge \frac{p}{1-p}$.

Remark. If $n \le \frac{1}{b-a}\left(a+b+2b\frac{p}{1-p}\right)$, then the distribution of X_n is strictly unimodal. In fact, in this case $(n-1)\frac{\beta}{\alpha}-(\alpha-1)2\left(\frac{\beta}{\alpha}+1\right)\le 0$ and $\{A_l\}_{l \ge n}$ changes sign at most once.

Acknowledgement. The author is grateful to Professor K. Sato for his guidance. He suggested the author to investigate time evolution of the Lévy process considered in this paper.

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