On the Kloosterman-sum Zeta-function

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The aim of the present paper is to show that given the spectral resolution of the hyperbolic Laplacian there is an argument which leads us fairly quickly to Kuznetsov's spectral expansion [1, (7.26)] of the Kloosterman-sum zeta-function $Z_{m,n}(s)$. Since his formula is equivalent to his much quoted trace formula [1, Theorem 2], our argument provides the latter with a more accessible proof. To prove his result on $Z_{m,n}(s)$ Kuznetsov developed a quite ingenious argument of transforming the inner-product of two Poincaré series into a series of J-Bessel functions integrated with respect to their orders, and applied various averaging technique to extract the defining series of $Z_{m,n}(s)$. Thus, though powerful and impressive, his argument inevitably depended heavily on the theory of Bessel functions as is well indicated by his use of exotic identities such as that of Gegenbauer [3, p. 138 (1)]. We shall dispense with those heavy machineries altogether.

Before starting our discussion it should be worth remarking that though we restrict ourselves to the case of the full modular group $\Gamma = SL(2, \mathbf{Z})$ it is apparent that we do not lose any generality.

Now, the Kloosterman-sum zeta-function is defined as

$$Z_{m,n}(s) = (2\pi\sqrt{mn})^{2s-1} \sum_{l=1}^{\infty} S(m, n; l) l^{-2s},$$
$$\left(m, n > 0, \operatorname{Re}(s) = \sigma > \frac{3}{4}\right),$$

where $S(m, n; l)$ is the Kloosterman sum

$$\sum_{\substack{h=1 \\ (h,l)=1}}^{l} \exp(2\pi i(mh + n\bar{h})/l), \; h\bar{h} \equiv 1 \bmod l.$$

We are going to extract this series from Poincaré series. To this end we take the same initial step as Kuznetsov's or rather that of Selberg [2]. Thus, we introduce the Poincaré series

$$(1) \quad P_m(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\operatorname{Im}\gamma(z))^s \exp(2\pi i m\gamma(z)), \; \sigma > 1,$$

where m is a positive integer, $z = x + iy$ $(y > 0)$ and Γ_∞ the stabilizer of the point at infinity. We have the well-known Fourier expansion

$$(2) \quad \begin{aligned} P_m(z, s) = {} & y^s \exp(2\pi imz) + \\ & y^{1-s} \sum_{n=-\infty}^{\infty} \exp(2\pi inx) \sum_{l=1}^{\infty} l^{-2s} S(m, n; l) \\ & \times \int_{-\infty}^{\infty} \exp\left(-2\pi iny\xi - \frac{2\pi m}{l^2 y(1 - i\xi)}\right) \\ & \times (1 + \xi^2)^{-s} d\xi, \end{aligned}$$

which is equivalent to regrouping the summands in (1) according to the double coset decomposition $\Gamma_\infty \backslash \Gamma / \Gamma_\infty$. Thus Weil's estimate for $S(m, n; l)$ yields that $P_m(z, s)$ is regular in the region $\sigma > \frac{3}{4}$, where we have also the bound $P_m(z, s) \ll y^{1-\sigma}$, providing y is not too small. This means in particular that $P_m(z, s)$ is in the Hilbert space $L^2(\mathscr{F}, d\mu)$ when $\sigma > \frac{3}{4}$; here $\mathscr{F}$ is the fundamental region of Γ and $d\mu$ the Poincaré metric as usual. We should note that Weil's bound for $S(m, n; l)$ is not mandatory but a bound like Estermann's classical estimate is sufficient for our purpose. At any event the above implies that we may apply the spectral decomposition to the inner product.

$$\langle P_m(\cdot, s_1), P_n(\cdot, \bar{s}_2)\rangle = \int_{\mathscr{F}} P_m(z, s_1) \overline{P_n(z, \bar{s}_2)} d\mu(z).$$

To state the decomposition we let $\left\{\lambda_j = \kappa_j^2 + \frac{1}{4}; \kappa_j > 0, j \geq 1\right\} \cup \{0\}$ stand for the discrete spectrum of the hyperbolic Laplacian acting on $L^2(\mathscr{F}, d\mu)$. Also let ψ_j be an eigen-form corresponding to λ_j so that it has the Fourier expansion

$$\psi_j(z) = \sqrt{y} \sum_{n \neq 0} \rho_j(n) K_{i\kappa_j}(2\pi |n| y) \exp(2\pi inx),$$

where K_ν is the K-Bessel function of order ν. We may assume that the set $\{\psi_j\}$ forms an orthonormal system. Then we have, for $\sigma_1, \sigma_2 > \frac{3}{4}$

$(\mathrm{Re}(s_\nu) = \sigma_\nu)$,

(3) $\langle P_m(\cdot, s_1), P_n(\cdot, \overline{s_2})\rangle$

$$= \frac{\pi}{\Gamma(s_1)\Gamma(s_2)} (4\pi\sqrt{mn})^{1-s_1-s_2}(n/m)^{\frac{1}{2}(s_1-s_2)}$$

$$\times \left\{\sum_{j=1}^{\infty} \overline{\rho_j(m)}\rho_j(n)\Theta(s_1, s_2; \kappa_j) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_{2ir}(m)\sigma_{2ir}(n)\cosh(\pi r)\Theta(s_1, s_2; r)}{(mn)^{ir}|\zeta(1+2ir)|^2} dr\right\}.$$

Here $\sigma_a(m)$ is the sum of the ath powers of divisors of m, $\zeta(s)$ the Riemann zeta-function and

$$\Theta(s_1, s_2; r) = \Gamma\left(s_1 - \frac{1}{2} + ir\right) \times \Gamma\left(s_1 - \frac{1}{2} - ir\right)\Gamma\left(s_2 - \frac{1}{2} + ir\right)\Gamma\left(s_2 - \frac{1}{2} - ir\right).$$

For the detailed proof of (3) see the relevant part of [1] although it is straightforward.

On the other hand the expression (2) and the unfolding device give

(4) $\langle P_m(\cdot, s_1), P_n(\cdot, \overline{s_2})\rangle$

$$= \delta_{m,n}\Gamma(s_1 + s_2 - 1)(4\pi m)^{1-s_1-s_2} + \sum_{l=1}^{\infty} l^{-2s_1}S(m, n; l) \times \int_{-\infty}^{\infty} (1+\xi^2)^{-s_1} Y_{s_2-s_1}(\xi; m, n, l)\,d\xi,$$

where $\delta_{m,n}$ is the Kronecker symbol and

$$Y_\omega(\xi; m, n, l) = \int_0^\infty y^{\omega-1} \times \exp\left(-2\pi ny(1+i\xi) - \frac{2\pi m}{l^2y(1-i\xi)}\right) dy.$$

To ensure the absolute convergence in (4) we impose the condition

(5) $$\sigma_2 > \sigma_1 > 1,$$

which is to be removed later.

So far we have followed Kuznetsov's argument. But we now depart from it. We are going to transform the integral in (4) into an expression similar to Barnes' integral representation of the hypergeometric function.

We have, by Mellin's formula,

$$Y_\omega(\xi; m, n, l) = \frac{1}{2\pi i}\int_0^\infty y^{\omega-1} \times \exp(-2\pi ny(1+i\xi))\int_{(\alpha)} \Gamma(\eta)\left(\frac{2\pi m}{l^2y(1-i\xi)}\right)^{-\eta} d\eta dy,$$

where (α) is the vertical line $\mathrm{Re}(\eta) = \alpha > 0$, and $|\mathrm{Arg}(1-i\xi)| < \frac{1}{2}\pi$. This double integral converges absolutely when

$$\mathrm{Re}(\omega) + \alpha > 0.$$

We exchange the order of integration and compute the inner integral, getting

$$Y_\omega(\xi; m, n, l) = \frac{1}{2\pi i}(2\pi n)^{-\omega} \times \int_{(\alpha)} \Gamma(\eta)\Gamma(\eta+\omega)\left(\frac{2\pi\sqrt{mn}}{l}\right)^{-2\eta} \frac{(1-i\xi)^{\eta}}{(1+i\xi)^{\eta+\omega}} d\eta,$$

where we have $|\mathrm{Arg}(1+i\xi)| < \frac{1}{2}\pi$. Thus, providing

(6) $$\alpha + \sigma_2 - \sigma_1 > 0,$$

we have, for the integral in (4),

$$\int_{-\infty}^{\infty} (1+\xi^2)^{-s_1} Y_{s_2-s_1}(\xi; m, n, l)\,d\xi = \frac{1}{2\pi i}(2\pi n)^{-\omega}\int_{-\infty}^{\infty}\int_{(\alpha)} \Gamma(\eta)\Gamma(\eta + s_2 - s_1) \times \left(\frac{2\pi\sqrt{mn}}{l}\right)^{-2\eta} \frac{(1-i\xi)^{\eta-s_1}}{(1+i\xi)^{\eta+s_2}} d\eta d\xi.$$

But, this double integral converges absolutely when

(7) $$\alpha - \sigma_1 + \frac{1}{2} < 0.$$

To see this it is enough to note that the integrand is, by Stirling's formula,

$$\ll |\xi|^{-\sigma_1-\sigma_2}|\eta|^{2\alpha-\sigma_1+\sigma_2-1} \times \exp(-\pi|\eta| + \mathrm{Im}(\eta)\mathrm{Arg}(1+i\xi) - \mathrm{Im}(\eta)\mathrm{Arg}(1-i\xi))$$
$$\ll |\xi|^{-\sigma_1-\sigma_2}|\eta|^{2\alpha-\sigma_1+\sigma_2-1}\exp(-|\eta|/|\xi|),$$

provided $|\eta|$ and $|\xi|$ are sufficiently large. Thus on (7) we may exchange the order of integration. In the resulting inner-integral we deform the contour and see that it is equal to

$$\int_C \frac{(1-i\xi)^{\eta-s_1}}{(1+i\xi)^{\eta+s_2}} d\xi,$$

where C starts at $+i\infty$, goes down along the imaginary axis, describes a small circle round i in the positive direction and returns to $+i\infty$. We note that $1 - i\xi$ is essentially positive and $\mathrm{Arg}(1 + i\xi)$ varies from $-\pi$ to π round the contour. We assume temporarily that $\alpha + \sigma_2 < 1$. Then the circular part of C can be collapsed to i, and the integral is equal to

$$(-ie^{\pi i(\eta+s_2)} + ie^{-\pi i(\eta+s_2)})\int_1^\infty \frac{(1+\xi)^{\eta-s_1}}{(\xi-1)^{\eta+s_2}} d\xi$$
$$= 2\sin(\pi(\eta+s_2))2^{1-s_1-s_2}\Gamma(1-\eta-s_2)\Gamma(s_1+s_2-1)/\Gamma(s_1-\eta)$$
$$= 2^{2-s_1-s_2}\pi\Gamma(s_1+s_2-1)/\{\Gamma(\eta+s_2)\Gamma(s_1-\eta)\}.$$

By the analytic continuation we may obviously drop the condition $\alpha + \sigma_2 < 1$.

Collecting these we now have

(8) $\langle P_m(\cdot, s_1), P_n(\cdot, \overline{s_2})\rangle =$

$$\delta_{m,n}\Gamma(s_1+s_2-1)(4\pi m)^{1-s_1-s_2} + 2^{2(1-s_2)}\pi^{s_1-s_2+1}n^{s_1-s_2}\Gamma(s_1+s_2-1)$$

$$\times \sum_{l=1}^{\infty} l^{-2s_1} S(m, n; l) W\left(\frac{4\pi\sqrt{mn}}{l}; s_1, s_2\right),$$

where

$$W(x; s_1, s_2) = \frac{1}{2\pi i} \int_{(\alpha)} \frac{\Gamma(\eta)\Gamma(\eta + s_2 - s_1)}{\Gamma(\eta + s_2)\Gamma(s_1 - \eta)} \left(\frac{x}{2}\right)^{-2\eta} d\eta.$$

Checking the convergence we see that (8) holds when

$$\sigma_2 + \alpha > \sigma_1 > \alpha + \frac{3}{4} \ (\alpha > 0).$$

The conditions (6) and (7) are fulfilled, and (5) has now been removed.

Now we specialize (8) by putting

$$s_1 = 1, \ s_2 = s$$

with

$$\mathrm{Re}(s) = \sigma > 1 - \alpha, \quad 0 < \alpha < \frac{1}{4}.$$

Note that

$$W(x; 1, s) = \frac{1}{2\pi i} \int_{(\alpha)} \frac{\Gamma(\eta)}{\Gamma(1 - \eta)(\eta + s - 1)} \left(\frac{x}{2}\right)^{-2\eta} d\eta.$$

We are going to transform this into a series of the Noumann type. For this sake we observe that for any integer k

$$\frac{\Gamma(k + s)\Gamma(k + \eta)}{\Gamma(k + 1 - s)\Gamma(k + 1 - \eta)} - \frac{\Gamma(k - 1 + s)\Gamma(k - 1 + \eta)}{\Gamma(k - s)\Gamma(k - \eta)}$$
$$= (s + \eta - 1)(2k - 1) \frac{\Gamma(k - 1 + s)\Gamma(k - 1 + \eta)}{\Gamma(k + 1 - s)\Gamma(k + 1 - \eta)}.$$

Thus we have, for any $K > 0$,

$$\frac{\Gamma(\eta)}{\Gamma(1 - \eta)(\eta + s - 1)} = \frac{\Gamma(1 - s)}{\Gamma(s)} \cdot \frac{\Gamma(K + s)\Gamma(K + \eta)}{\Gamma(K + 1 - s)\Gamma(K + 1 - \eta)(s + \eta - 1)}$$
$$- \frac{\Gamma(1 - s)}{\Gamma(s)} \sum_{k=1}^{K} (2k - 1) \times \frac{\Gamma(k - 1 + s)\Gamma(k - 1 + \eta)}{\Gamma(k + 1 - s)\Gamma(k + 1 - \eta)}.$$

This implies that

$$W(x; 1, s) = \frac{\Gamma(1 - s)}{2\pi i \Gamma(s)} \cdot \int_{(\alpha)} \frac{\Gamma(K + s)\Gamma(K + \eta)}{\Gamma(K + 1 - s)\Gamma(K + 1 - \eta)(\eta + s - 1)} \left(\frac{x}{2}\right)^{-2\eta} d\eta$$
$$- 2x^{-1} \frac{\Gamma(1 - s)}{\Gamma(s)} \sum_{k=1}^{K} (2k - 1) \frac{\Gamma(k - 1 + s)}{\Gamma(k + 1 - s)} J_{2k-1}(x),$$

where the last factor is the J-Bessel function of order $2k - 1$ (cf. [3, p. 192 (7)]). As K tends to infinity the last integral converges to

$$2\pi i \left(\frac{x}{2}\right)^{2(s-1)},$$

which is the residue of the pole at $1 - s$. To see this we move the contour to $(-A)$ with a large positive $A < K$, passing over $1 - s$, and note that the integrand of the resulting integral is, by Stirling's formula,

$$\ll_{A,s,x} K^{2\sigma - 1}(K + |\eta|)^{-2A-1},$$

which gives the assertion. Thus we find that

$$(9) \quad W(x; 1, s) = \frac{\Gamma(1 - s)}{\Gamma(s)} \left(\frac{x}{2}\right)^{2(s-1)} - 2x^{-1} \times \frac{\Gamma(1 - s)}{\Gamma(s)} \sum_{k=1}^{\infty} (2k - 1) \frac{\Gamma(k - 1 + s)}{\Gamma(k + 1 - s)} J_{2k-1}(x).$$

We then collect (3) with $s_1 = 1$, (8) and (9), finishing the proof of

Theorem (Kuznetsov [1]). *We have, for any positive* m, n *and* $\mathrm{Re}(s) > \frac{1}{2}$,

$$Z_{m,n}(s) = \frac{1}{2} \sin(\pi s) \sum_{j=1}^{\infty} \frac{\overline{\rho_j(m)\rho_j(n)}}{\cosh(\pi\kappa_j)} \times \Gamma\left(s - \frac{1}{2} + i\kappa_j\right)\Gamma\left(s - \frac{1}{2} - i\kappa_j\right)$$
$$+ \frac{1}{2\pi} \sin(\pi s) \int_{-\infty}^{\infty} \frac{\sigma_{2ir}(m)\sigma_{2ir}(n)}{(mn)^{ir} |\zeta(1 + 2ir)|^2} \times \Gamma\left(s - \frac{1}{2} + ir\right)\Gamma\left(s - \frac{1}{2} - ir\right) dr$$
$$+ \sum_{k=1}^{\infty} p_{m,n}(k) \frac{\Gamma(k - 1 + s)}{\Gamma(k + 1 - s)} - \frac{1}{2\pi} \delta_{m,n} \frac{\Gamma(s)}{\Gamma(1 - s)},$$

where

$$p_{m,n}(k) = (2k - 1) \sum_{l=1}^{\infty} \frac{1}{l} S(m, n; l) J_{2k-1}\left(\frac{4\pi\sqrt{mn}}{l}\right).$$

It should be noted that Petersson's famous formula expresses $p_{m,n}(k)$ in terms of Fourier coefficients of holomorphic cusp-forms of weight $2k$.

References

[1] N. V. Kuznetsov: Petersson hypothesis for parabolic forms of weight zero and Linnik hypothesis. Sums of Kloosterman sums. Mat. Sbornik, **111**, 334–383 (1980) (Russian).

[2] A. Selberg: On the estimation of Fourier coefficients of modular forms. Proc. Symp. Pure Math., 8, 1–15 (1965).

[3] G. N. Watson: A Treatise on the Theory of Bessel Functions. Cambridge Univ. Press (1944).