Finding Singular Support of a Function from its Tomographic Data

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Abstract: We develop three different approaches to the problem of finding discontinuity surface S of a function $f(x)$ from its tomographic data: 1) pseudolocal tomography, introduced by the authors, produces S and the jumps of f across S from pseudolocal tomographic data; 2) local tomography, introduced in the literature and further generalized by one of the authors produces S from local data. A method is proposed for finding jumps of f across S. This method is based on the analysis of pseudodifferential operators acting on piecewise smooth functions; and 3) geometric tomography, introduced by one of the authors and A. Zaslavsky, produces S from $\hat{S}$, a dual variety with respect to S, which happens to be the singular support of $\hat{f}(\theta, p)$.

I. Introduction. Let $f(x)$ be a compactly supported piecewise-smooth function. For simplicity we assume that $x \in \boldsymbol{R}^2$, but the basic ideas and results are valid in $\boldsymbol{R}^n$, $n \geq 2$. Define the Radon transform $f(\theta, p) := \int_{\boldsymbol{R}^2} f(x)\delta(p - \Theta \cdot x)dx$, where δ is the delta-function, $\Theta \in S^1$, S^1 is the unit sphere in $\boldsymbol{R}^2$, $\Theta \cdot x$ is the dot product, $p \in \boldsymbol{R}^1$. Here and everywhere below we use the scalar variable θ, $0 \leq \theta < 2\pi$, along with the vector variable Θ so that $\Theta = (\cos\theta, \sin\theta)$. The Radon transform has been studied in [2, 4, 14]. In many applications one is interested in finding discontinuity surfaces of $f(x)$ given $\hat{f}(\theta, p)$. The traditional way to do it is to invert the Radon transform. This is called the standard tomography. The inversion formula is well known [14]. This formula requires integration with respect to θ and p, and therefore is an expensive operation. One is interested in finding a fast and less computationally expensive procedure for finding discontinuity surfaces S of f.

We describe three different approaches to this problem: 1) pseudolocal tomography (PLT), introduced by the authors in [6] and further developed in [13], produces the discontinuity surfaces S of f and jumps of f across S from pseudolocal tomographic data; 2) local tomography (LT), introduced in [31] and [30] and further generalized in [17], produces S from local data, but no methods for finding jumps of f across S were given in the literature. We develop a new method for finding these jumps within the framework of local tomography. Also, we give an approach to optimizing the LT formulas [22]; and 3) geometric tomography, introduced in [24], produces S from $\hat{S}$, a dual variety with respect to S, which happens to be the singular support of $\hat{f}(\theta, p)$.

A fast algorithm to recover S is given by local tomography [31, 1]. This procedure is to calculate $(-\Delta)^{1/2}f$. From ellipticity of $-\Delta$ it follows [3] that $(-\Delta)^{1/2}f$ and f have the same singular support, and one can prove that calculation of $(-\Delta)^{1/2}f$ at a point x given $\hat{f}(\theta, p)$ can be done using only the intergrals of f along straight lines passing through the point x. This results in an inversion formula which uses integration with respect to θ-variable only. Local tomography, as developed in [31] and [1], produces the image of the discontinuity surface S of f, but it does not give the values of jumps of f across S. These jumps are of practical importance in many applications.

In this announcement we present the following new results: 1) we introduce the concept of pseudolocal tomography, construct a formula which allows one to calculate fast the discontinuity curve S of f given the PLT data. By the PLT data we mean the knowledge of the integrals

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of f over the lines which intersect a disc of small radius $d > 0$ centered at a point x. The PLT formula also allows one to calculate values of jumps of f across S ; 2) we construct a family of local tomography functions and prove that these functions have the same wave fronts as f ; in particular, they have the same singular supports as f. We describe asymptotic behavior of the function $(Bf)(x)$ in a neighborhood of a point $x_0 \in S$ (the discontinuity of f), where B is a pseudodifferential operator (PDO) with a symbol satisfying some natural conditions which are met in many cases. Using this result, we obtain a formula which allows one to compute values of jumps of f across S within the framework of local tomography; 3) we identify the set $\hat{S}$ in the space (θ, p) which is in a one-to-one correspondence with S, the singular support of f, and construct the map that sends $\hat{S}$ onto S. This yields the third efficient way to calculate S from tomographic data. A systematic study of the singular support of $\hat{f}(\theta, p)$ and its relation to S is given in [24-27].

In Section II the results are formulated. In Section III a brief outline of some of the proofs is given. Related papers are [6-11], [16], [17-22], [24-28], and [29].

II. Basic results. 1. *Pseudolocal tomography.* Let us assume for simplicity that f is a compactly supported real-valued piecewise-continuous function, supp $f = D \subset \boldsymbol{R}^2$, and let $S \subset D$ be the discontinuity curve of f. We suppose that S is a piecewise smooth oriented curve, S_+ will denote the part of D which is on the positive side of S near S and S_- is the part of D which is on the negative side of $S, f \in C^\infty(D \setminus S)$ unless other assumptions are stated explicitly. The smoothness assumptions can be relaxed, but we do not go into detail (see [6]). Define the *pseudolocal tomography function* f_d as follows:

$$f_d(x) := (4\pi)^{-2} \int_{S^1} \int_{\Theta\cdot x-d}^{\Theta\cdot x+d} \hat{f}_p(\theta, p)[\Theta\cdot x - p]^{-1} dp d\theta, \; x \in \boldsymbol{R}^2, \tag{1}$$

where $d > 0, \hat{f}_p := \partial \hat{f}/\partial p$, and define the complementary function $f_d^c(x) := f(x) - f_d(x)$.

Clearly, calculation of $f_d(x)$ at a point x involves integration over only such (θ, p) that $|\Theta\cdot x - p| \leq d$, i.e., it requires the knowledge of integrals of f along lines intersecting a disk centered at x with radius d. Thus, to speed up calculations, we have to take d in (1) as small as possible. Therefore we need to investigate the properties of f_d as $d \to 0$. Since $f_d^c := f - f_d$, the above problem reduces to the investigation of the convergence $f_d^c \to f$ as $d - 0$. Moreover, since calculation of f_d^c does not involve integration over the interval $[x\cdot\Theta - d, x\cdot\Theta + d]$ where the Cauchy kernel is singular (this follows immediately from the well-known inversion formula for the Radon transform), the removal of the interval can be considered as a possible method of regularizing the inversion formula. Therefore, the convergence $f_d^c \to f$ as $d \to 0$ can be considered as convergence of a regularized convolution and backprojection (CB) algorithm to the original density function. General results on convergence of CB algorithms which take into account both regularization (which is different from ours) and discretization are obtained in [15].

In Theorem 1 we investigate the convergence $f_d^c \to f$ in three cases: a) on compact sets not intersecting S, b) at the points of S, and c) in a neighborhood of S. In particular, we establish the existence of a layer of width $O(d)$ around S inside which f_d^c does not converge to f in the *sup*-norm. Additional results on convergence $f_d^c \to f$ are formulated in Theorem 4 below. Let $U \subset \boldsymbol{R}^2$ be an open set, $\bar{U}$ be its closure.

Theorem 1. *If $\bar{U} \cap S = \emptyset$ and $x \in \bar{U} \subset D$ then $|f_d^c(x) - f(x)| = O(d)$ as $d \to 0$ and convergence is uniform in $\bar{U}$. If $x_0 \in S$ and there exists a neighborhood U of the point x_0 such that S is smooth in U, then $\left| f_d^c(x_0) - \frac{f_+(x_0) + f_-(x_0)}{2} \right| = O(d\,|\ln d|)$ as $d \to 0$, where $f_+(x_0)$ and $f_-(x_0)$ are the limiting values of $f(x)$ as x approaches x_0 from S_+ and S_-, respectively. Let n_+ be a unit normal to S pointing into S_+, $n_- = -n_+$, $\gamma > 0$ is a fixed number, $D_\pm(x_0) = f_\mp(x_0) - f_\pm(x_0)$. Then*

$$\lim_{d\to 0} f_d(x_0 + \gamma d n_\pm) = \lim_{d\to 0} [f(x_0 + \gamma d n_\pm) - f_d^c(x_0 + \gamma d n_\pm)] = -D_\pm(x_0)\phi(\gamma), \tag{2}$$

$$\phi(\gamma) = \frac{2}{\pi^2} \int_0^{\min(1,1/\gamma)} \frac{\arccos(\gamma t)}{(1-t^2)^{1/2}} dt, \; \gamma > 0.$$

Moreover, $\phi(\gamma) > 0$ is monotonically decreasing, $\phi(+0) = 0.5$, and $\phi(\gamma) = 2(\pi^2\gamma)^{-1} + O(\gamma^{-3})$ as $\gamma \to \infty$.

Let $U_d := \{x : x \in \boldsymbol{R}^2, \text{dist}(x, U) \leq d\}$, where U is an open set in $\boldsymbol{R}^2$, $B(x_0, d) := \{x :$

$|x-x_0| \le d\}$, $A(x_0, a, b) := \{x : a \le |x-x_0| \le b\}$, $0 \le a \le b$, $\partial B(x_0, d) := \{x : |x-x_0| = d\}$.

Theorem 2. *Fix $x_0 \in \boldsymbol{R}^2$ such that either $\partial B(x_0, d) \cap S = \phi$ or $\partial B(x_0, d)$ is transversal to S at any point $\tilde{x}$ of $\partial B(x_0, d) \cap S$, and S is C^∞ in a neighborhood of $\partial B(x_0, d) \cap S$. Then $f_d^c(x) \in C^\infty$ in a neighborhood of x_0.*

Corollary (Pseudolocality property). $f \in C^\infty(U_d) \Rightarrow f_d \in C^\infty(U)$.

Let k and m denote integers $k, m = 0,1,2, \dots$, and D^k denote any partial derivative of order k.

Remark. In fact, if $f \in C^k(U_d)$, $k \ge 0$, then $f_d \in C^k(U)$.

Theorem 3. (*a*) *If $f \in C^{k-1}(U_d)$, $k \ge 1$, $D^k f$ exists and has a jump across S, S is piecewise-smooth in U_d, then $D^k f_d^c \in C(U)$.* (*b*) *If $k = 0$ and $f(x)$ has finite limits as x approaches S from S_+ and S_-, then $f_d^c \in C(U)$.*

Remark. Theorem 3 describes "preservation of discontinuities". Theorem 2 and 3 tell us, in particular, that the difference $f_d^c = f - f_d$ is a continuous function if f is piecewise continuous, hence f_d and f have the same discontinuity curve S and the same jumps across S. Thus Theorem 3 is the basis for the PLT. More details on its numerical realization and results of numerical experiments are given in [6].

Theorem 4. *If $f \in C^m(B(x_0, R))$, then one has for $x \in B(x_0, R)$ as $d \to 0$:*

$|D^k f_d^c(x) - D^k f(x)| = o(1)$ *if* $k = m$

$|D^k f_d^c(x) - D^k f(x)| = o(d|\log d|)$ *if* $k = m - 1$

$|D^k f_d^c(x) - D^k f(x)| = O(d)$ *if* $k \le m - 2$.

The estimates are uniform in $x \in B(x_0, R')$ for any fixed R', $R' < R$.

The generalization of (1) is a *family of PLT functions* $f_{\sigma d}$:

(3) $$f_{\sigma d}(x) := \frac{1}{4\pi^2}\int_{S^1}\int_{\Theta\cdot x-d}^{\Theta\cdot x+d} \frac{\sigma_d(\Theta\cdot x - p)}{\Theta\cdot x - p}\hat{f}_p(\theta, p)\,dp\,d\theta,$$

where $\sigma_d(p)$ satisfies the properties

(1) $\sigma_d(p)$ is real-valued and even;

(2) $\sigma_d(p)$ is piecewise continuously differentiable and there are at most finitely many points at which σ_d is discontinuous; and

(3) $\sigma_d(p) = \sigma_1(p/d)$, $|\sigma_d(p) - 1| \le O(p)$, $p \to 0$.

Remark. Taking $\sigma_d(p) \equiv 1$ in (3), we get the PLT function defined in (1).

Theorem 5. *The difference $f - f_{\sigma d}$ is continuous, hence the functions f and $f_{\sigma d}$ have the same discontinuity curve S and the same values of jumps across S.*

Although Theorem 5 asserts that the jumps of f and $f_{\sigma d}$ are identical, this is not sufficient for practical purposes. More precisely, we need to know the behavior of $f_{\sigma d}$ in a neighborhood of S, that is, we need to generalize eq. (2) for the PLT function $f_{\sigma d}$. This is done in Section II.2 below (see eq. (10)) using the relation between PLT and LT functions. More details about the family of PLT functions (3) and results of numerical experiments are presented in [13].

2. *Local tomograhy.* Let

(4) $$b(\Theta) \in C^\infty(S^1), \quad \min_{\Theta\in S^1} b(\Theta) > 0.$$

Define

(5) $$B(x) := -\frac{1}{4\pi}\int_{S^1} b(\Theta)\hat{f}_{pp}(\theta, \Theta\cdot x)\,d\theta.$$

Let $WF(f)$ denote the wave front of f.

Theorem 6. *If* (4) *holds then* $WF(B) = WF(f)$.

Corollary. *sing supp* (B) = *sing supp* (f).

Equation (5) is an LT formula for any $b(\Theta)$ which satisfies conditions (4). The LT formula of [1] is obtained from (5) by taking $b(\Theta) \equiv 1$. Note that eq. (5) can be written as

(6) $$B(x) = \mathscr{F}^{-1}\{|\xi| b(\Theta)\hat{f}(\xi)\},$$
$$\Theta := \xi|\xi|^{-1}, \ \hat{f}(\xi) = \mathscr{F}f,$$

where $\mathscr{F}$ is the Fourier transform. As it was noted, the LT function $B(x)$ does not preserve jumps of f across the discontinuity curve S. However the values of jumps can be recovered using the following result.

Theorem 7. *Let the assumption be the same as in Theorem* 1. *Let us pick any $x_0 \in S$ such that S is smooth in a neighborhood U of x_0. Let us consider the action of the PDO with symbol $g(\xi)$ on f*:

(7) $$(Bf)(x) = \int_{\boldsymbol{R}^2}\int_{\boldsymbol{R}^2} g(\xi)\exp(i\xi(x-y))f(y)\,dy\,d\xi,$$

where $g(\xi)$ satisfies the conditions $g(\xi) = |\xi|^\gamma b(\xi|\xi|^{-1})(1 + O(|\xi|^{-1}))$ as $|\xi| \to \infty$ for some $\gamma > 0$, $b(\Theta)$ as in (4), *and $b(\Theta) = b(-\Theta)$. Then*

(8) $$(Bf)(x_s + hn_s) = \frac{b(n_s)D(x_s)}{\pi}\sqrt{\frac{R(x_s)}{R(x_s)+h}}\, Im\left\{\int_0^\infty \psi_{\gamma-1}(t, x_s)\exp(ith)\,dt\right\} + \eta(x_s + hn_s), \quad x_s \in S \cap U, \ h \to 0,$$

where $D(x_s) = \lim_{t\to 0+}[f(x_s + tn_s) - f(x_s - tn_s)]$, *$R(x_s)$ and n_s are the radius of curvature of S and the unit vector perpendicular to S at x_s, respectively,*

$\psi_{\gamma-1}$ is some function such that $\psi_{\gamma-1}(t, x_s) \in C^\infty([0, \infty) \times (S \cap U))$ and $\psi_{\gamma-1}(t, x_s) = t^{\gamma-1}(1 + O(t^{-1}))$, $t \to \infty$, where $O(t^{-1})$ is uniform for $x_s \in S \cap K$ for any compact K, $x_0 \in K \subset U$, and $\eta \in C^\infty(U)$.

Remark. The Radon transform is an even function of (θ, p) and the assumption $b(\Theta) = b(-\Theta)$ simplified the resulting formula (8). Because of the lack of space, we do not give here the general formula for the case $b(\Theta) \neq b(-\Theta)$.

Taking $\gamma = 1$ in eq. (8), using the well-known identity $\int_0^\infty \exp(ith)dt = \pi\delta(h) + i/h$ and eqs. (6) and (7), we get

Corollary. *One has*

(8′) $$B(x_s + hn_s) = \frac{b(n_s)D(x_s)}{\pi} h^{-1}(1 + o(1)),\ h \to 0.$$

Formula (8) and (8′) imply that the behavior of $B(x)$ in a neighborhood of $x_s \in S$ depends only on the magnitude $D(x_s)$ of a jump of f across S at x_s. Thus, one can recover $D(x_s)$ using (8′).

We have developed also an approach for choosing an optimal formula (5) which gives the most noise-stable estimate of $B(x)$ by the criterion of the minimum variance of the error [22]. It is assumed that $\hat{f}(\theta, p)$ is observed with noise $n(\theta, p)$, and under certain assumptions about the noise, the optimal $b(\Theta)$ is found from the requirement ($*$) $\int_{S^1}\int_{S^1} b(\Theta)b(\Theta')R(\theta, \theta')d\theta d\theta' = \min$ under the constraints (4) and $(2\pi)^{-1}\int_{S^1} b(\Theta)d\theta = 1$. Here $R(\theta, \theta')$ is the covariance matrix of $n_{pp}(\theta, p)$ calculated at $(\theta, p = \Theta \cdot x)$ and $(\theta', p' = \Theta' \cdot x)$ and integrated with respect to x over $\boldsymbol{R}^2$ against a certain weight function $w(x) > 0$, $\int_{\boldsymbol{R}^2} w(x)dx = 1$. If $R(\theta, \theta') = R(|\theta - \theta'|)$, then ($*$) has the unique solution $b(\Theta) = 1$. In such a case, the standard LT formula with $b(\Theta) = 1$ is optimal. In general, $b(\Theta) \neq \text{const}$.

Let f_Λ denote the standard LT function (5) with $b(\Theta) \equiv 1$. The reason for using such notation is clear from (4), because $f_\Lambda = \Lambda f = (-\Delta)^{1/2}f$, where $(-\Delta)^{1/2}$ is the square root of the negative Laplacian. It is interesting that there exists a close relation between the family of PLT functions (3) and f_Λ. Indeed, let a family of functions $\sigma_d(p)$ satisfying conditions (a)-(c) from Section II.1 (below (3)) be given. Then, choosing a radial function $M(x)$ such that its Radon transform satisfies

(9) $$\hat{M}'_{\sigma d}(p) := \frac{\sigma_d(p)}{\pi p},\ |p| \le d,\quad \hat{M}_{\sigma d}(-d) = \hat{M}_{\sigma d}(d) = 0,$$

that is $\hat{M}_{\sigma d}(p) = -\int_{-d}^{p} \frac{\sigma_d(s)}{\pi s}ds$, $|p| \le d$, we obtain $f_{\sigma d} = M_{\sigma d} * f_\Lambda$ [13]. Using this relation and eq. (8′), we derive [13]:

$$f_{\sigma d}(x_s + hn_s) = (M_{\sigma d} * f_\Lambda)(x_s + hn_s) = D(x_s)\psi_{\sigma d}(h) + O(d\ \mathrm{In}^2 d),\ \frac{|h|}{d} < \infty,\ d \to 0,$$

(10) $$\psi_{\sigma d}(h) = \frac{1}{\pi^2}\int_{-d}^{d} \mathrm{In}\,|h - t|\,\frac{\sigma_d(t)}{t}dt.$$

Letting $\sigma_d(t) = 1$, $|t| \le d$, in (10) one can show that eqs. (10) and (2) are in complete agreement.

3. *Geometric tomography. Duality law and the map $\hat{S} \to S$.* In this section we will find a subset $\hat{S}$ in the space (θ, p) which is in a one-to-one correspondence with the discontinuity surfaces S of f, and the map which sends $\hat{S}$ onto S. It turns out that the set $\hat{S}$ is *singsupp* $\hat{f}(\theta, p)$, and there is a simple geometrical connection between S and $\hat{S}$ which we call the duality law: *both S and $\hat{S}$ are envelopes of a one-parametric family of straight lines tangent to each of them for appropriate choices of parameters.* Thus, if one wishes to find S given $\hat{f}(\theta, p)$, one can find $\hat{S}$ first (e. g., by the methods given in [7, 10]) and then map it to S by the Legendre transform as explained below. Practical applications of these ideas are given in [7].

Let ($*$) $x_2 = g(x_1)$ be the equation of S in local coordinates (x_1, x_2). Assume $g \in C^2$. Define $q = -p/\theta_2$, $\beta := -\theta_1/\theta_2$ for $\theta_2 \neq 0$, and let ($**$) $q = h(\beta)$ be the equation of $\hat{S}$ in the local coordinates (β, q). Here $\hat{S}$ is the dual variety with respect to S, that is, $\hat{S}$ is a set of (θ, p) such that the line $\Theta \cdot x = p$ is not transversal to S for some $x \in S$. Define the Legendre transform Lg of the function $g(x_1)$: consider the equation $g'(x_1) = \beta$, assume that this equation is uniquely solvable for $\beta \in U_{\bar\beta}$, where $U_{\bar\beta}$ is a neighborhood of the point $\bar\beta = g'(\bar x_1)$; then $Lg := \beta x_1(\beta) - g(x_1(\beta))$. The notion of the Legendre transform is classically defined for functions $C^2(U)$, $U \subset \boldsymbol{R}^n$, in which case $Lg := \beta \cdot x(\beta) -$

$g(x(\beta))$, $\beta \in \boldsymbol{R}^n$, $\beta \cdot x$ is the dot product. A generalized Legendre transform is defined in [25, 27]. One can prove that $Lg = h(\beta)$, where $g(x_1)$ and $h(\beta)$ are the functions in (∗) and (∗∗). Since L is involutive, i.e. $L^2 = I$, where I is the identity operator, one obtains $g = Lh$. This is the recipe for calculating S given $\hat{S}$: *if* $h(\beta)$ *is known, then the unknown* $g(x_1)$ *can be found by the formula* $g = Lh$.

Let us explain the above statement concerning envelopes. Consider the family of straight lines

(11) $$\beta x_1 - q - x_2 = 0.$$

Take $q = h(\beta)$, so that lines (11) are tangent to S, and consider (11) as a one-parametric family of lines, β being the parameter. Then the envelope of (11) is S, and the equation of S in the local coordinates is obtained by classical rule for calculation of the envelopes. This yields the equation of S of the form $x_2 = g(x_1)$, where $g = Lh$, L is the Legendre transform. On the other hand, take $x_2 = g(x_1)$ in (11), where $x_2 = g(x_1)$ is the local equation of S, and consider (11) is a family of straight lines in the space (β, q). Then $\hat{S}$ is the envelope of this family of lines and the equation of $\hat{S}$ is $q = h(\beta)$, where $h(\beta) = Lg$ (see [25, 27]).

III. Outline of some of the proofs. 1. One can prove that $f_d(x) = -\pi^{-1}_{\varepsilon\to+0}\lim\int_\varepsilon^d F_q(x, q)$ $q^{-1}dq$ and

$$f_d^c(x) = -\pi^{-1}\int_d^\infty F_d(x, q)q^{-1}dq$$
$$= \bar{f}(0, x) - \frac{2d}{\pi}\int_d^\infty [f(r, x) - \bar{f}(0, x)]r^{-2}[1 - (d/r)]^{-1/2}dr.$$

Here $F_q := \partial F/\partial q$, $F = (2\pi)^{-1}\int_{S^1}\hat{f}(\theta, x\cdot\Theta + q)d\theta$, and $\bar{f}(r, x) := (2\pi)^{-1}\int_{S^1}f(x + r\Theta)d\theta$, $\bar{f}(0, x) := \bar{f}(+0, x)$. Pick any x_0 and $R > 0$ such that $f(x) \in C^2(B(x_0, R))$. Denoting $M_0(x_0) := \sup_{x\in\boldsymbol{R}^2}|f(x) - f(x_0)|$, $M_2(r, x_0) := \sup_{\substack{x,i,j \\ |x - x_0|\le r}} |\partial^2 f(x)/\partial x_i\partial x_j|$, we obtain

(12) $$|f_d^c(x_0) - f(x_0)| \le \frac{2d}{\pi}\int_d^R \frac{M_2(r, x_0)dr}{[1 - (d/r)^2]^{1/2}} + \frac{2d}{\pi}\int_R^\infty \frac{|\bar{f}(r, x_0) - \bar{f}(0, x_0)|}{[1 - (d/r)^2]^{1/2}}\frac{dr}{r^2}$$
$$\le \frac{2d}{\pi}[RM_2(R, x_0) + M_0(x_0)R^{-1}] + O(d^3R^{-3}) \text{ as } d\to 0.$$

Now we pick $x_0 \in S$ such that S is smooth in a neighborhood of x_0. We denote by n_+ the unit normal to S at x_0 pointing into S_+ and $n_- = -n_+$. Let $f_\pm(x_0)(\nabla f_\pm(x_0))$ be the limits of $f(x)(\nabla f(x))$ as $x \to x_0$ from S_+ or S_-. Note that $\bar{f}(0, x_0) = [f_-(x_0) + f_+(x_0)]/2$. Let $D_\pm(x_0) := f_\mp(x_0) - f_\pm(x_0)$, $R_0 > 0$ be the radius of curvature of S at the point x_0, and $A := D_+(x_0)(2\pi R_0)^{-1} + [\nabla f_+(x_0)\cdot n_+ + \nabla f_-(x_0)\cdot n_-](4\pi)^{-1}$. Approximating $f(x)$ on each side of S using $f_\pm(x_0)$ and $\nabla f_\pm(x_0)$, one can prove that

(13) $$|f_d^c(x_0) - \bar{f}(0, x_0)| \le \frac{2d}{\pi}|A|\ln\frac{2R}{d} + O(d) \text{ as } d\to 0,$$

(14) $$\lim_{d\to 0}[f(x_0 + \gamma d n_\pm) - f_d^c(x_0 + \gamma d n_\pm)] = -D_\pm(x_0)\left\{\frac{2}{\pi^2}\int_0^{\min(1,1/\gamma)}\frac{\arccos(\gamma t)}{(1 - t^2)^{1/2}}dt\right\}, \gamma > 0.$$

Theorem 1 follows from estimates (12)-(14).

Changing variables in the definition of f_d^c, we get

$$f_d^c(x) = \int_{|y-x|>d}\mu(y - x)f(y)\,dy,$$
$$\mu(y) := d\pi^{-2}|y|^{-3}[1 - d^2|y|^{-2}]^{-1/2}.$$

This formula allows one to prove that if $f = 0$ on U_d then $f_d^c(x) \in C^\infty(U)$. This and some additional estimates lead to Theorems 2-4. Statement (b) of Theorem 3 can also be proved by considering the difference between f_d (see eq. (1)) and a similar inversion formula for f from $\hat{f}$, integrating by parts in p, and by using that $\hat{f}(\theta, p)$ is discontinuous, at most, at countably many pairs (θ, p). Theorem 5 is proved analogously.

2. Theorem 6 is proved using formula (6). The operator $Q : f(x) \mapsto B(x)$ has the property $WF(Qf) = WF(f)$. This follows from the PDO theory [3]: the symbol $b(\Theta)$ is an elliptic PDO of order zero, and $|\xi|^m$ is a symbol of a PDO which preserves wavefront. Thus, the operator Q preserves wavefront of f. To prove Theorem 7, we pick any $x_0 \in S$ such that S is smooth in a neighborhood of x_0. Let $w(x)$ be a C_0^∞ cut-off function with a sufficiently small support and $w = 1$ in a neighborhood of x_0. Denote $f_w = fw$. Clearly, $Bf - Bf_w \in C^\infty$ in a neighborhood of x_0. Choose the coordinate system such that the origin coincides with the center of curvature of S

at x_0, and the x_1-axis is perpendicular to S at x_0. The center of curvature is supposed to be on a positive side of S, i.e. S_+. Consider the rays through the origin which intersect S inside the support of w. Let $D_w(\theta)$ denote the jump of f_w at the intersection of S and the ray having angle θ with the x_1-axis, and $p_0(\theta)$ denote the distance from the origin to the straight line which is tangent to S and is perpendicular to the above ray. Using the Fourier slice theorem [14, p. 11] and the results from [26], we get from (7) in the above coordinate system

$$B(x) - \frac{2}{(2\pi)^2}\mathrm{Re}\left\{\Gamma(1.5)e^{-3i\pi/4}\int_0^\infty\int_{-\theta_1}^{\theta_2} g(t\Theta)2\sqrt{2R(\theta)}\,D_w(\theta)\psi_{-1.5}(t,\theta)e^{it(p_0(\theta)-\Theta\cdot x)}d\theta t dt\right\} \in C^\infty.$$

Using assumption from Theorem 7 about the asymptotic of the symbol g, $g(t\Theta) \sim t^\gamma b(\Theta)$, $t\to\infty$, substituting $x = x_s + hn_s$, into the above equation, investigating properties of the function $a(\theta, x) = p_0(\theta) - \Theta\cdot x$, and using the stationary phase method [3], we prove Theorem 7.

3. The ideas and results from Sections II.1 and II.2 can be generalized to n-dimensional problems, $n > 2$, and to more general cases of the Radon transform, e. g. exponential Radon transform [5].

4. Description of the singular support of the Radon transform as a dual variety to the singular support of f can be generalized to X-ray transform of f [27].

5. The results and techniques used in this paper are based on the papers [6-13, 17-26].

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