On Factors of Automorphy of Fractional Weight for Cocompact Fuchsian Groups

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§1. Introduction. We set $G = SL(2, \mathbf{R})$ and fix a positive integer $l \geq 2$. Let $\tilde{G}$ be the *l-fold* covering group of G. We have an exact sequence

(1) $$1 \to \mu_l \to \tilde{G} \to G \to 1$$

where μ_l is the cyclic group of order l. By (1), we obtain the cohomology class $\xi \in H^2(G, \mu_l)$, which is of order l. For a subgroup Γ of G, let ξ_Γ be the restriction of ξ to Γ. The purpose of this paper is to determine when ξ_Γ splits, in the case where Γ is a cocompact torsion free subgroup of G.

We can construct $\tilde{G}$ as follows ([4]). For $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $z \in \mathbf{H}$ (the complex upper half plane), put $j(\sigma, z) = cz + d$. Let

$$\tilde{G} := \{(\sigma, f(\sigma, z)) \mid \sigma \in G, f \text{ is holomorphic on } \mathbf{H}, f^l(\sigma, z) = j(\sigma, z)\}.$$

We define the group law of $\tilde{G}$ by

$$(\sigma, f(\sigma, z))(\tau, g(\tau, z)) = (\sigma\tau, f(\sigma, \tau z)g(\tau, z)).$$

By this definition of $\tilde{G}$, we see easily that ξ_Γ splits if and only if there exists an automorphic factor of Γ whose l-*th* power is equal to $j(\sigma, z)$.

For example, if l is even and Γ contains -1, then ξ_Γ does not split because there is no automorphic factor as above.

However if $l = 2$ and $\Gamma = \Gamma(4)$ (the principal congruence subgroup of $SL(2, \mathbf{Z})$ of level 4), then ξ_Γ splits by the automorphic factor $h(\sigma, z) = \theta(\sigma(z))/\theta(z)$ $(\sigma \in \Gamma)$. Here $\theta(z) = \sum_{n=-\infty}^{\infty} \exp(2\pi\sqrt{-1}\, n^2 z)$ $(z \in \mathbf{H})$ is the standard theta function. (See [3].)

The main result of this paper is:

Theorem. *Let Γ be a discrete torsion-free subgroup of G such that $\Gamma \backslash G$ is compact. Let g be the genus of $\Gamma \backslash \mathbf{H}$. Let*

(2) $$1 \to \mu_l \to \tilde{\Gamma} \to \Gamma \to 1$$

be the exact sequence obtained from ξ_Γ.

Then (2) *splits if and only if* $g \equiv 1 \pmod{l}$.

Moreover the order of ξ_Γ is $l/(l, g-1)$, where $(l, g-1)$ denotes the greatest common divisor of l and $g-1$.

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§2. Poincaré series. When a non-vanishing holomorphic function $h(\sigma, z)$ of $z \in \mathbf{H}$ is given for every $\sigma \in \Gamma$ which satisfies

$$h(\sigma_1\sigma_2, z) = h(\sigma_1, \sigma_2 z)h(\sigma_2, z)$$

for every $\sigma_1, \sigma_2 \in \Gamma$ and $z \in \mathbf{H}$, we call $h(\sigma, z)$ an *automorphic factor* with respect to Γ.

Fix $k \in \mathbf{R}$ and let Γ be a torsion free subgroup of G such that $\Gamma \backslash G$ is compact and $h(\sigma, z)$ be an automorphic factor with respect to Γ. We consider a meromorphic function $f(z)$ on $\mathbf{H}$ satisfying the following conditions.

(i) f has finitely many poles.

(ii) Let $\{z_1, \ldots, z_m\}$ be the set of poles of f. For any neighborhood U_u of $z_u (1 \leq u \leq m)$, we have

$$\int_{\mathbf{H}'} | f(z) y^{\frac{k}{2l}} | \, y^{-2} dxdy < \infty$$

where $\mathbf{H}' = \mathbf{H} - \cup_{u=1}^{m} U_u$ and $z = x + \sqrt{-1}y$.

Using this f, we can construct a Poincaré series

$$F(z) = \sum_{\gamma \in \Gamma} \frac{f(\gamma z)}{h(\gamma, z)^k}.$$

Since we can see formally that $F(\sigma z) = h(\sigma, z)^k F(z)$ for any $\sigma \in \Gamma$, F is a meromorphic form of weight k with respect to $h(\sigma, z)$ if the Poincaré series uniformly converges on any compact subset outside of the poles.

Proposition (see [1] p. 64). (1) *Poincaré series $F(z)$ converges absolutely uniformly on any compact subset of $\mathbf{H} - \{\gamma z_u \mid \gamma \in \Gamma, 1 \leq u \leq m\}$. Moreover $F(z)$ is meromorphic on $\mathbf{H}$.*

(2) *If $f(z)$ has a pole of order r at w and holomorphic at γw for every $\gamma \in \Gamma - \{1\}$, then $F(z)$ has a pole of order r at w.*

Take any integer k such that $k \geq 4l$, $(k, l) = 1$; take any integer n such that $\frac{k}{2l} + 1 < n$

$\leqq \frac{k}{l}$. Then we can apply the proposition to $f(z) = (z - i)^{-n}$ and obtain the following corollary.

Corollary. *Γ being as above, assume that there exists an automorphic factor $h(\sigma, z)$ with respect to Γ whose l-th power is equal to $j(\sigma, z)$. Then for any $k \geq 4l$ such that $(k, l) = 1$, there exists a non-zero meromorphic function F which satisfies $F(\sigma z) = F(z)h(\sigma, z)^k$ for $\sigma \in \Gamma$, $z \in \mathbf{H}$.*

§3. Proof of Theorem. Now we will complete the proof of the theorem.

First assume that (2) splits. Then there exists an automorphic factor $h(\sigma, z)$ with respect to Γ whose l-th power is equal to $j(\sigma, z)$. Take any integer $k \geq 4l$ such that $(k, l) = 1$. By the corollary we get a non-zero meromorphic form F of weight k with respect to the automorphic factor $h(\sigma, z)$. It is clear that F^l is a meromorphic form of weight k in the usual sense. Then we see that

$$\deg(\mathrm{div}(F^l)) = k(g - 1).$$

(See [2], p. 39, Prop. 2. 16.)

Since the left hand side is divisible by l and k is prime to l, we get $g \equiv 1 \pmod{l}$.

Conversely let $g = ln + 1$. Take a non-zero meromorphic differential 1-form ω on the Riemann surface $\mathfrak{R} := \Gamma \backslash \mathbf{H}$. Then

$$\deg(\mathrm{div}(\omega)) = 2g - 2 = 2ln.$$

Take a point P_0 of $\mathfrak{R}$. Then

$$\deg(\mathrm{div}(\omega) - 2lnP_0) = 0.$$

By Jacobi's inversion problem, the divisor group whose element is of degree zero is divisible modulo linear equivalence. Hence we can take a divisor B on $\mathfrak{R}$ such that

$$\mathrm{div}(\omega) - 2lnP_0 \sim 2l\mathrm{B},$$

where $\sim$ means the linear equivalence. By definition of linear equivalence, there exists a meromorphic function f on $\mathfrak{R}$ such that

$$2l\mathrm{B} - \mathrm{div}(\omega) + 2lnP_0 = \mathrm{div}(f).$$

In view of the correspondence between meromorphic differential 1-forms and meromorphic forms of weight two, we get a moromorphic form F of weight two such that

$$Fdz = f\omega.$$

We get

$$\mathrm{div}(F) = \mathrm{div}(f\omega) \quad \text{(See [2], p. 39, Prop. 2. 16.)}$$
$$= \mathrm{div}(f) + \mathrm{div}(\omega) = 2l(\mathrm{B} + nP_0).$$

Therefore there exists a meromorphic function H such that $H^{2l} = F$. Set $h'(\sigma, z) = \frac{H(\sigma z)}{H(z)}$. Then we have $h'(\sigma, z)^{2l} = j(\sigma, z)^2$. From this, we see easily that $h'(\sigma, z)^l = \chi(\sigma)j(\sigma, z)$ with a homomorphism χ of Γ to $\{\pm 1\}$. Since Γ is isomorphic to the fundamental group of a compact Riemann surface of genus ≥ 2, there is a character χ_0 of Γ such that $\chi_0^l = \chi$. Now $h(\sigma, z) = \frac{h'(\sigma, z)}{\chi_0(\sigma)}$ is the automorphic factor which splits (2).

Let $m = l/(l, g - 1)$ and m' be the order of ξ_Γ. We can take a subgroup Γ' of Γ of index m. Let $\mathbf{Res} : H^2(\Gamma, \mu_l) \to H^2(\Gamma', \mu_l)$ be the restriction map, and $\mathbf{Cor} : H^2(\Gamma', \mu_l) \to H^2(\Gamma, \mu_l)$ be the corestriction map. Let g' be the genus of $\Gamma' \backslash \mathbf{H}$. By the Hurwitz formula, we have $g' - 1 = m(g - 1)$. Hence we have already shown $\mathbf{Res}(\xi_\Gamma) = \xi_{\Gamma'} = 1$. Since $\mathbf{Cor} \circ \mathbf{Res}(\xi_\Gamma) = \xi_\Gamma^m = 1$, m is divisible by m'.

Let $f : \mu_l \to \mu_{l/m'}$ be the m'-th power map, and $f_* : H^2(\Gamma, \mu_l) \to H^2(\Gamma, \mu_{l/m'})$ be the induced map by f. Since $f_*(\xi_\Gamma) = \xi_\Gamma^{m'} = 1$, $g - 1$ is divisible by $\frac{l}{m'}$. Hence m' is divisible by m.

References

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