The Rankin's L-function and Heegner Points for General Discriminants

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In this paper we show some results on the relation between the first derivatives of the Rankin's L-series of certain modular forms at $s = 1$ and the heights for certain divisors on the Jacobian of the modular curve $X_0(N)$. These divisors consist of Heegner points whose orders have conductor f. The proof of our main result consists of a long complicated "analytic" computation (see [5]). This generalizes the "analytic" part of the influential work of Gross and Zagier [4], which has established a relation between the first derivatives of the Rankin's L-series of certain modular forms at $s = 1$ and the height pairings for squarefree discriminants prime to N. Their results can be applied to give the proof of a special case of the Birch-Swinnerton-Dyer conjecture, and are needed to complete Goldfeld's solution of Gauss conjecture for the class number of imaginary quadratic fields. Kolyvagin [6] has used the result of [4] in his proof of the finiteness of the Tate-Shafarevich groups of certain modular elliptic curves over $\boldsymbol{Q}$. J. van der Lingen [7] has calculated "algebraically" the local Néron-Tate height pairings "at non-archimedean places" for certain divisor on $X_0(N)$ consisting of Heegner points whose orders have general discriminants prime to N. He has found explicit formulas for these local height pairings at non-archimedian places. But it is difficult to compare his formulas and ours.

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§1. Let us begin with recalling some definitions. Let K be an imaginary quadratic field with the fundamental discriminant D_0, and $\mathcal{O}$ an order with the discriminant $D = D_0 f^2$ of the conductor f in K. Let $h_f = \# \operatorname{Pic}(\mathcal{O})$ and $u = \# (\mathcal{O}^\times / \{\pm 1\})$. We have $u = 1$ unless $D = -3, -4$, in which cases $u = 3, 2$, respectively.

We say $x = (E \to E')$ is a "Heegner point" of discriminant D on $X_0(N)$ if both of the elliptic curves E and E' have complex multiplication by $\mathcal{O}$. Such a point exists if and only if D is congruent to a square modulo $4N$; equivalently every prime divisor of N splits or is ramified in K. If one Heegner point exists on $X_0(N)$, then there are $2^s \cdot h_f$ Heegner points with $s = \# \{p \mid N\}$ which are all rational over the "ring class field" $K_f = K(j(E))$ of K. Those Heegner points are attached to a fixed integral ideal $\mathfrak{n}(\mathrm{N}(\mathfrak{n}) = N)$ of $\mathcal{O}$ with $\mathcal{O}/\mathfrak{n} \xrightarrow{\sim} \boldsymbol{Z}/N\boldsymbol{Z}$. They are permuted simply-transitively by the abelian group $W \times \operatorname{Gal}(K_f/K)$ and those actions on Heegner points can be described explicitly, where $W \cong (\boldsymbol{Z}/2\boldsymbol{Z})^s$ is the group of Atkin-Lehner involutions and $\operatorname{Gal}(K_f/K)$ the Galois group of K_f/K, which is canonically isomorphic to the class group $\operatorname{Pic}(\mathcal{O})$ of $\mathcal{O}$ via the Artin reciprocity map (see [1]).

In this paper, D is not assumed to be square free nor relatively prime to N on $X_0(N)$, but assume throughout that the conductor f is relatively prime to N. Fix a Heegner point x of discriminant D; then the class of the divisor $c = (x) - (\infty)$ defines an element in $J(K_f)$, where (∞) denotes the sum of cusps at infinity on $X_0(N)$, which is defined over $\boldsymbol{Q}$, where J is the Jacobian of $X_0(N)$.

Let $f(z) = \sum_{n \geq 1} a(n) e^{2\pi i n z}$ be an element in the vector space of newforms of weight 2 on $\Gamma_0(N)$, $\varepsilon(\cdot) = \left(\frac{D}{\cdot}\right)$ the Kronecker Symbol and $r_{\mathscr{A}}(n)$ the number of integral invertible ideals of $\mathcal{O}$ of norm n in the class $\mathscr{A}$. We define the Rankin's L-function associated to the newform $f(z)$ and the ideal class $\mathscr{A}$ by

$$L_{\mathscr{A}}(f, s) = L^{(N)}(2s - 2k + 1, \varepsilon) \cdot \sum_{n \geq 1} a(n) r_{\mathscr{A}}(n) n^{-s},$$

where

$$L^{(N)}(2s - 2k + 1, \varepsilon) = \sum_{\substack{n \geq 1 \\ (n, DN) = 1}} \varepsilon(n) n^{-2s+2k-1}.$$

The series $L^{(N)}$ is the Dirichlet L-function of ε at the argument $2s-2k+1$ without the Euler factor at all primes dividing N.

For an eigenform $f(z)$ of the action of the Hecke algebra $\mathbf{T}$ normalized by the condition that $a_1=1$, and a complex character χ of the ideal classgroup of $\mathcal{O}$, we define the L-function by

$$L(f,\chi,s)=\sum_{\mathscr{A}}\chi(\mathscr{A})L_{\mathscr{A}}(f,s).$$

§2. Choose a Heegner point x of discriminant $D=D_0f^2$ on $X_0(N)$ (D_0; fundamental discriminant). We assume that the conductor f is prime to N, squarefree, odd, $(D_0,f)=1$ and $\mathrm{lcd}(D,N)=N'|D|$ with $N'=\prod_{p^t\|N,\ p\nmid D}p^t$.

Let c be the class of the divisor $(x)-(\infty)$ in $J(K_f)$. The element σ in the Galois group of K_f/K corresponds to the ideal class $\mathscr{A}$ of $\mathrm{Pic}(\mathcal{O})$ under the Artin isomorphism. Let $\langle\ ,\ \rangle_\infty$ denote the local height pairing at infinity on $J(K_f)\otimes\boldsymbol{Q}$, $\langle\ ,\ \rangle_p$ at the prime p and $(\ ,\ )$ the Petersson inner product on cusp forms of weight 2 for $\Gamma_0(N)$. Finally, we let $f(z)$ be a new form of weight 2 on $\Gamma_0(N)$ and T_m the m-th Hecke correspondence. Then we have:

Theorem 1. 1) The function $L_{\mathscr{A}}(f,s)$ and $L(f,\chi,s)$ have analytic continuations to the entire s-plane, satisfy for $L^*_{\mathscr{A}}(f,s):=(2\pi)^{-2s}N'^s|D|^s\Gamma(s)^2L_{\mathscr{A}}(f,s)$ functional equation

$$L^*_{\mathscr{A}}(f,s)=-\varepsilon(N')L^*_{\mathscr{A}}(f,2-s)$$

and vanish at the point $s=1$.

2) The series $g_{\mathscr{A}}(z)=\sum_{m\geq1}\langle c,T_mc^\sigma\rangle e^{2\pi imz}$ is a cusp form of weight 2 on $\Gamma_0(N)$, whose Fourier coefficients are given by

$$\langle c,T_mc^\sigma\rangle=\langle c,T_mc^\sigma\rangle_\infty+\sum_p\langle c,T_mc^\sigma\rangle_p$$

with

$$\langle c,T_mc^\sigma\rangle_\infty\in\mathbf{R},\ \langle c,T_mc^\sigma\rangle_p\in\mathbf{Z}\log p,$$
$$\langle c,T_mc^\sigma\rangle_p=0 \text{ for almost all } p.$$

3) There is a cuspform $\Phi_{\mathscr{A}}(z)=\sum_{m=1}^\infty a_me^{2\pi imz}$ such that

i) $(f,\Phi_{\mathscr{A}})=\dfrac{u^2\sqrt{|D|}}{8\pi^2}L'_{\mathscr{A}}(f,1)$ for all newforms $f(z)$ of weight 2 on $\Gamma_0(N)$,

ii) $a_m=a_{m,\infty}+\sum_p a_{m,p}$ with $a_{m,\infty}\in\mathbf{R}$, $a_{m,p}\in\mathbf{Z}\log p$ (for all p) and $a_{m,p}=0$ for almost all p,

iii) $a_{m,\infty}=\langle c,T_mc^\sigma\rangle_\infty$.

Remark. It follows from parts 2) and 3) of the theorem, that $\langle c,T_mc^\sigma\rangle$ and a_m differ by the logarithm of a rational number for every m, or equivalently that the cusp forms $\Phi_{\mathscr{A}}$ and $g_{\mathscr{A}}$ differ by a cusp form (of weight 2 and level N) all of whose Fourier coefficients are logarithms of natural numbers. In view of the finite-dimensionality of the space of cusp forms of fixed weight and level and the linear independence over $\mathbf{Q}$ of the logarithms of prime numbers, this shows that there are only finitely many primes p for which the equality $a_{m,p}=\langle c,T_mc^\sigma\rangle_p$ fails for any m. In fact, of course, we conjecture that there are no such primes, i.e.:

Conjecture 1. We have $a_{m,p}=\langle c,T_mc^\sigma\rangle_p$ for every p.

Remark. In view of the results in 2) and 3), this conjecture is equivalent to $a_m=\langle c,T_mc^\sigma\rangle$ or to $\Phi_{\mathscr{A}}=g_{\mathscr{A}}$. It is simply the analogue of the Gross-Zagier result under our weaker assumptions, and would (or will) be a consequence of Theorem 1 as soon as the local height calculations at finite primes are carried out in this generality.

We can also consider a corresponding result for the first derivatives $L'(f,\chi,s)$ as Gross-Zagier, where f is a normalized eigenform and χ is a complex character of the class group $\mathrm{Pic}(\mathcal{O})$. We identify χ with a character of $\mathrm{Gal}(K_f/K)$, and define $c_\chi=\sum_\sigma\chi^{-1}(\sigma)c^\sigma$ in the χ-eigenspace of $J(K_f)\otimes\mathbf{C}$ (This is h_f times the standard eigencomponent). So by using the bilinearity of the global height pairing, we can derive from Theorem 1 and conjecture 1 through a purely formal calculation

Theorem 2. Let $c_{\chi,f}$ be the projection of c_χ to the f-isotypical component of $J(K_f)\otimes\mathbf{C}$ under the action of $\mathbf{T}$. Assume that Conjecture 1 hold. Then we have

$$L'(f,\chi,1)=\frac{8\pi^2(f,f)}{h_fu^2|D|^{1/2}}\hat{h}(c_{\chi,f}),$$

i.e.

$$L'(f,\chi,1)=\frac{\|\omega_f\|^2}{u^2|D|^{1/2}}\hat{h}(c_{\chi,f}),$$

where $\omega_f=2\pi if(z)dz$ is the eigendifferential associated to $f(z)$, $\|\omega_f\|=\iint_{X_0(N)(\mathbf{C})}\omega_f\wedge i\bar{\omega}_f=8\pi^2(f,f)$ and the quadratic form $\hat{h}$ is the canonical Néron-Tate height associated to the class of the divisor $2(\Theta)$ with symmetric theta-divisor Θ in $J(K_f)$.

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