A Recurrence Formula for the Bernoulli Numbers

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1. The theorem. Let B_n $(n = 0,1,2,\ldots)$ be the Bernoulli numbers defined by the formal power series

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

and put $\tilde{B}_n = (n+1)B_n$. As is well known and easily seen, $\tilde{B}_1 = -1$ and $\tilde{B}_n = 0$ for all odd integers ≥ 3. In this note we present the following recurrence relation.

Theorem. *The $\tilde{B}_n$'s satisfy*

$$\tilde{B}_{2n} = -\frac{1}{n+1}\sum_{i=0}^{n-1}\binom{n+1}{i}\tilde{B}_{n+i} \quad (n \geq 1). \tag{1}$$

Remark. The formula has a strong resemblance to the usual recurrence

$$B_n = -\frac{1}{n+1}\sum_{i=0}^{n-1}\binom{n+1}{i}B_i$$

(see [2] for example) but needs half the number of terms to calculate B_{2n}.

We shall give two proofs. The first proof uses a continued fraction expansion and its convergents of the defining power series of B_{2n}. This method faithfully traces our original way of discovering the formula and seems to apply to searching similar kinds of formulas for various numbers defined by nice generating functions. The second and much simpler proof is due to Don Zagier, to whom the author expresses his gratitude for permitting him to include the proof in the paper.

2. Convergents of continued fraction expansion. Let $f(x) = 1 + c_1x + c_2x^2 + \cdots$ be a formal power series (over some field) with constant term 1. Suppose $f(x)$ has a continued fraction expansion

$$f(x) = \frac{1}{1+}\,\frac{a_1x}{1+}\,\frac{a_2x}{1+}\,\frac{a_3x}{1+}\cdots \tag{2}$$

with non-zero a_i's and let

$$\frac{Q_n(x)}{P_n(x)} = \frac{1}{1+}\,\frac{a_1x}{1+}\cdots\frac{a_{n-1}x}{1+a_nx}$$

be its n-th convergent. The polynomials $P_n(x)$ and $Q_n(x)$ are uniquely determined from $f(x)$ by the following conditions:

$$P_n(0) = Q_n(0) = 1. \tag{3}$$

(4) $\deg P_n(x) = \deg Q_n(x) = m$ if $n = 2m$, $\deg P_n(x) = \deg Q_n(x) + 1 = m + 1$ if $n = 2m+1$

$$f(x) \equiv Q_n(x)/P_n(x) \bmod x^{n+1} \tag{5}$$

(in the ring of formal power series).

Both $P_n(x)$ and $Q_n(x)$ satisfy the same recurrence relations

$$P_n(x) = P_{n-1}(x) + a_nxP_{n-2}(x), \tag{6}$$
$$Q_n(x) = Q_{n-1}(x) + a_nxQ_{n-2}(x) \quad (n \geq 2)$$

with the initial conditions $P_0 = 1$, $P_1 = 1 + a_1x$, $Q_0 = Q_1 = 1$.

Now we put $f(x) = (\sqrt{x}/2)\coth(\sqrt{x}/2)$, where $\coth y = (e^y + e^{-y})/(e^y - e^{-y})$. This is a generating function of even index Bernoulli numbers:

$$f(x) = \sum_{n=0}^{\infty} B_{2n}\frac{x^n}{(2n)!}.$$

In this case, the coefficients a_i in (2) are given by $a_1 = -1/12$, $a_{2n} = \binom{2n+2}{2}\Big/\left(12\binom{4n+1}{4}\right)$ and $a_{2n+1} = \binom{2n}{2}\Big/\left(12\binom{4n+4}{4}\right)$ $(n \geq 1)$. This can be deduced from the famous expansion

$$\frac{\tanh\sqrt{x}}{\sqrt{x}} = \frac{1}{1+}\,\frac{x}{3+}\,\frac{x}{5+}\cdots$$

with the aid of a formula for the inverse of a given continued fraction expansion ([3, p. 332]), but we omit the details here. The key point of our proof of the theorem lies in the explicit description of the convergents of the continued fraction expansion of $f(x)$ $(= (\sqrt{x}/2)\coth(\sqrt{x}/2))$.

Lemma. *With the notations as above, we have*

$$P_{2m}(x) = \sum_{i=0}^{m}\binom{2m}{2i}\binom{4m+1}{2i}^{-1}\frac{x^i}{(2i+1)!} \quad (m \geq 0)$$

$$P_{2m-1}(x) = \frac{2}{m(4m^2-1)}\sum_{i=0}^{m}(2m-2i-1)(2m^2+i) \times \binom{2m+1}{2i+1}\binom{4m}{2i+1}^{-1}\frac{x^i}{(2i+1)!} \quad (m \geq 1)$$

$$Q_{2m}(x) = \sum_{i=0}^{m}\binom{2m+1}{2i}\binom{4m+2}{2i}^{-1}\frac{x^i}{(2i)!} \quad (m \geq 0)$$

$$Q_{2m-1}(x) = \frac{1}{m(4m^2-1)} \sum_{i=0}^{m-1} (m-i)(4m^2+2i-1) \binom{2m+1}{2i} \binom{4m}{2i}^{-1} \frac{x^i}{(2i)!} \; (m \geq 1).$$

Proof. E. Heine [1, p. 245] gave the convergents of the continued fraction expansion of $1/f(x)$. Taking the properties (4) and (5) of the convergents into account, we see that the $2m$-th convergent for $1/f(x)$ is just the inverse of that for $f(x)$, i.e. $P_{2m}(x)/Q_{2m}(x)$, thus we obtain the formula for $P_{2m}(x)$ and $Q_{2m}(x)$. Thanks to the recurrence (6), odd index P's and Q's are calculated from even index ones and the lemma follows.

3. Proof of the theorem. By the approximation property (5), we have

$$(7) \quad \left(\sum_{i=0}^{2n} B_{2i} \frac{x^i}{(2i)!}\right) P_{2n}(x) \equiv Q_{2n}(x) \bmod x^{2n+1}$$

and

$$(8) \quad \left(\sum_{i=0}^{2n-1} B_{2i} \frac{x^i}{(2i)!}\right) P_{2n-1}(x) \equiv Q_{2n-1}(x) \bmod x^{2n}.$$

Equating the coefficients of x^{2n}, x^{2n-1} of (7) and x^{2n-1} of (8) by using Lemma, we get respectively

$$(9) \quad \tilde{B}_{4n} = -\frac{1}{2n+1} \sum_{i=0}^{n-1} \binom{2n+1}{2i} \tilde{B}_{2n+2i} \; (n \geq 1)$$

$$(10) \quad \tilde{B}_{4n-2} = -\frac{1}{4n(2n+1)(4n+1)} \sum_{i=0}^{n-1} (2n+2i+1)(2n+2i) \binom{2n+1}{2i} \tilde{B}_{2n+2i-2} \; (n \geq 2)$$

$$(11) \quad \tilde{B}_{4n-2} = -\frac{1}{2n^2(4n^2-1)} \sum_{i=0}^{n-1} (2i-1) \times (2n^2+n-i) \binom{2n+1}{2i} \tilde{B}_{2n+2i-2} \; (n \geq 2).$$

Multiplying (10) by $4n(2n+1)(4n+1)$, (11) by $4n^2(4n^2-1)$ and adding them give us

$$(12) \quad \tilde{B}_{4n-2} = -\frac{1}{2n} \sum_{i=1}^{n-1} \binom{2n}{2i-1} \tilde{B}_{2n+2i-2} \; (n \geq 2)$$

or

$$(13) \quad \tilde{B}_{4n+2} = -\frac{1}{2n+2} \sum_{i=0}^{n-1} \binom{2n+2}{2i+1} \times \tilde{B}_{2n+2i+2} \; (n \geq 1).$$

We can unify (9), (13) and $\tilde{B}_2 = 1/2$ into (1) in the theorem (recall that $\tilde{B}_1 = -1$ and $\tilde{B}_{\text{odd} \geq 3} = 0$), hence completes the proof.

4. Another proof. The simple proof sketched below is due to D. Zagier.

In general, define an involution $*$ on the set of sequences $\{b_0, b_1, b_2, \ldots\}$ by

$$B^*(x) = e^{-x} B(-x),$$

where $B(x)$ is the following generating function:

$$B(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{(n+1)!}.$$

$\left(\text{i.e., by } b_n^* = (-1)^n \sum_{i=0}^{n} \binom{n+1}{i+1} b_i.\right)$ Then the expression

$$\sum_{i=0}^{n} \binom{n}{i} b_{n+i-1} \; (n \geq 1)$$

is seen to be anti-invariant under $*$ and hence vanishes if $B^*(x) = B(x)$. This is the case when $B(x) = x/(e^x - 1)$, thus we have

$$\sum_{i=0}^{n} \binom{n}{i} \tilde{B}_{n+i-1} = 0.$$

Replacing n by $n+1$ and observing $\tilde{B}_{2n+1} = 0$, we get the theorem.

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References

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