Wavelet Transforms Associated to a Principal Series Representation of Semisimple Lie Groups. II

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1. Introduction. Let G be a noncompact connected semisimple Lie group with finite center and $P = MAN$ a parabolic subgroup of G. Let $\pi_\lambda = \mathrm{Ind}_P^G(1 \otimes e^\lambda \otimes 1)$ $(\lambda \in \mathfrak{a}_C^*)$ denote a principal series representation of G and $(\pi_\lambda, L^2(\bar{N}, e^{-2\Re\lambda(H(\bar{n}))}d\bar{n})$ $(\bar{N} = \theta(N))$ the noncompact picture of π_λ. Let σ_ω denote an irreducible unitary representation of $\bar{N}$ corresponding to $\omega \in \bar{\mathfrak{n}}_C^*$ and (S, ds) a subset of MA with measure ds. In the previous paper [3] we supposed that there exists a $\phi \in \mathcal{S}'(\bar{N})$ satisfying the following admissible condition: for all $\omega \in V_T'$

(i) $\sigma_\omega(\phi)\sigma_\omega(\phi)^* = n_\phi(\omega)I$,

(ii) $0 < \int_S n_\phi(Ad(s)\omega)\,ds = c_{S,\phi} < \infty$,

where $c_{S,\phi}$ is independent of ω (see [3] for the notations). Then for all such ϕ we can deduce the inversion formula:

$$f(x) = c_{S,\phi}^{-1} \int\int_{\bar{N}\times S} \langle f, \pi_{-i\rho}(\bar{n}s)\phi \rangle \cdot \pi_{-i\rho}(\bar{n}s)\phi(x)\,d\bar{n}ds \quad \text{for all } f \in \mathcal{S}(\bar{N}),$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $L^2(\bar{N})$. A number of well-known examples of wavelet transforms arises from this scheme through the explicit form of ϕ. However, in the case of $G = SL(n+2, \boldsymbol{R})$ $(n \geq 1)$ and $\bar{N} \cong H_n$, the $(2n+1)$-dimensional Heisenberg group, the above formula does not cover the three examples constructed by Kalisa and Torrésani (see [4, IV]). Therefore, in order to obtain a widespread application we need to generalize this formula. In this paper we suppose that S is an arbitrary measurable set with map $l: S \to G$ and then we shall consider a distribution vector ϕ in $\mathcal{S}'(\bar{N})$ which depends on $s \in S$.

2. Main theorem. We retain the notations in [3] except that (S, ds) is an arbitrary measurable set with map $l: S \to G$. Let Ψ be a family of $\phi_s \in \mathcal{S}'(\bar{N})$ with parameter $s \in S$. We call the quartet $\mathfrak{A} = (\lambda, S, l, \Psi)$ satisfies the admissible condition if for all $\omega \in V_T'$ and $F \in L^2(\boldsymbol{R}^k)$

$$\int_S \sigma_\omega(\pi_\lambda(l(s)\phi_s))\sigma_\omega(\pi_\lambda(l(s)\phi_s))^* F\,ds = c_{\mathfrak{A}}F,$$

where σ_ω is realized on $L^2(\boldsymbol{R}^k)$ (see §3) and $c_{\mathfrak{A}}$ is independent of ω.

Theorem 1. *Let $\mathfrak{A} = (\lambda, S, l, \Psi)$ satisfy the admissible condition. Then,*

$$f(x) = c_{\mathfrak{A}}^{-1} \int\int_{\bar{N}\times S} \langle f, \pi_\lambda(\bar{n}l(s))\phi_s \rangle \cdot \pi_\lambda(\bar{n}l(s))\phi_s(x)\,d\bar{n}ds \text{ for all } f \in \mathcal{S}(\bar{N}).$$

Proof. As shown in [2] it is enough to prove that

$$\int_S \| \langle f, \pi_\lambda(\cdot)\Psi_s \rangle \|_{L^2(\bar{N})}^2 ds = c_{\mathfrak{A}} \| f \|_{L^2(\bar{N})}^2,$$

where $\Psi_s = \pi_\lambda(l(s))\phi_s$. Since $\sigma_\omega(\langle f, \pi_\lambda(\cdot)\Psi_s \rangle) = \sigma_\omega(f)\sigma_\omega(\Psi_s)^*$, it follows from the Plancherel formula for $L^2(\bar{N})$ that

$$\begin{aligned} &\int_S \| \langle f, \pi_\lambda(\cdot)\phi_s \rangle \|_{L^2(\bar{N})}^2 ds \\ &= \int_S \int_{V_T'} \| \sigma_\omega(f)\sigma_\omega(\Psi_s)^* \|_{HS}^2 \mu(\omega)\,d\omega ds \\ &= \int_{V_T'} \mathrm{tr}\Big(\sigma_\omega(f) \int_S \sigma_\omega(\Psi_s)^* \sigma_\omega(\Psi_s)\,ds\,\sigma_\omega(f)^*\Big)\mu(\omega)\,d\omega \\ &= c_{\mathfrak{A}} \| f \|_{L^2(\bar{N})}^2. \end{aligned}$$

□

3. Admissible condition. In what follows we assume that

(A0) $l(S) \subset MA$,

and we shall obtain a sufficient condition of $\mathfrak{A} = (\lambda, S, l, \Psi)$ under which $\mathfrak{A}$ is admissible. Let $\mathfrak{q}$ be a polarizing subalgebra for all $\omega \in V_T'$ and Q the corresponding analytic subgroup of $\bar{N}$. We put $k = \mathrm{codim}\mathfrak{q}$, $\chi_\omega(\exp Y) = e^{2\pi i\omega(Y)}$ $(Y \in \mathfrak{q})$, and $\bar{n} = \exp X(\bar{n})\gamma(t(\bar{n}))$ $(X(\bar{n}) \in \mathfrak{q},\ t(\bar{n}) \in \boldsymbol{R}^k)$ where $\gamma: \boldsymbol{R}^k \to \bar{N}$ is a cross-section for $Q\backslash\bar{N}$. Then $\sigma_\omega = \mathrm{Ind}_Q^{\bar{N}}(\chi_\omega)$ and it is realized on $L^2(\boldsymbol{R}^k)$ as $\sigma_\omega(\bar{n})F(t) = \chi_\omega(X(\gamma(t)\bar{n}))F(t(\gamma(t)\bar{n}))$ (cf. [1, p.125]). Here we recall that $l(s) \in MA$ and a weak Malcev basis consists of root vectors for (G, A). Thus $Ad(l(s))$ stabilizes Q and $Q\backslash\bar{N}$ respectively. Here we suppose that

(A1) $\mathfrak{q}$ is ideal,

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(A2) $\psi_s(q\gamma(t)) = \psi(q)\chi_{\omega(s)}(q)\delta(t)\Delta(s)$
$(q \in Q,\ t \in \boldsymbol{R}^k)$,

where δ is the Dirac function on $\boldsymbol{R}^k$. For each $s \in S$, $q \in Q$, t, $t_0 \in \boldsymbol{R}^k$ it follows that $\gamma(t_0) Ad(l(s))(q\gamma(t)) = Ad(\gamma(t_0)l(s))q \cdot \gamma(t_0)Ad(l(s))\gamma(t)$ where $Ad(\gamma(t_0)l(s))q \in Q$ and $\gamma(t_0)Ad(l(s))\gamma(t) = \gamma(t(s, t, t_0))$ for some $t(s, t, t_0) \in \boldsymbol{R}^k$. Then for $F \in L^2(\boldsymbol{R}^k)$

$$\begin{aligned}
&\sigma_\omega(\pi_\lambda(l(s))\psi_s)F(t_0)\\
&= \int_{\bar{N}} \psi_s(l(s)^{-1}\bar{n})\sigma_\omega(\bar{n})F(t_0)d\bar{n}\\
&= e^{(i\lambda+\rho)\log l(s)} \int_{\bar{N}} \psi_s(Ad(l(s)^{-1})\bar{n})\sigma_\omega(\bar{n})F(t_0)d\bar{n}\\
&= e^{(i\lambda-\rho)\log l(s)} \int_{\bar{N}} \psi_s(\bar{n})\sigma_\omega(Ad(l(s))\bar{n})F(t_0)d\bar{n}\\
&= e^{(i\lambda-\rho)\log l(s)}\Delta(s) \int_Q \psi(q)\chi_{\omega(s)}(q)\\
&\qquad \chi_\omega(Ad(\gamma(t_0)l(s))q)dq \int_{\boldsymbol{R}^k} \delta(t)F(t(s, t, t_0))dt\\
&= e^{(i\lambda-\rho)\log l(s)}\Delta(s)\ \hat{\psi}(Ad^*(\gamma(t_0)l(s))\omega + \omega(s))F(t_0),
\end{aligned}$$

where $\log l(s) = \log a_s$ if $l(s) = m_s a_s \in MA$. Therefore, we can deduce that $\sigma_\omega(\pi_\lambda(l(s))\psi_s) \cdot \sigma_\omega(\pi_\lambda(l(s))\psi_s)^*$ is the multiplication operator on $L^2(\boldsymbol{R}^k)$ corresponding to

$$m_{\lambda,\omega,s}(t) = e^{-2(\Im\lambda+\rho)\log l(s)}\,|\Delta(s)|^2 \cdot |\hat{\psi}(Ad^*(\gamma(t)l(s))\omega + \omega(s))|^2.$$

Next we identify $\mathfrak{q}^*$ with $\boldsymbol{R}^m$ ($m = \dim \mathfrak{q}$) and define the (m, m)-matrix $L(s)$ by

$$Ad^*(l(s))X = L(s)X \quad (X \in \mathfrak{q}^*).$$

We assume the following,

(A3) there exist a measurable set (U, du) for which

$$S = U \times \boldsymbol{R}^m \text{ and } ds = dudx,$$

(A4) there exist (m, m)-matrices $A(s)$, $C_j(u)$ for which

(a) $\dfrac{\partial L(s)^{-1}}{\partial x_j} = A(s)C_j(u) \quad (1 \le j \le m)$,

(A5) $\omega(s) = L(s)h(s)$ $(h(s) \in \boldsymbol{R}^m)$ and there exist $d_j(u) \in \boldsymbol{R}^m$ such that

(b) $\dfrac{\partial h(s)}{\partial x_j} = A(s)d_j(u) \quad (1 \le j \le m)$,

(A6)
$$e^{-2(\Im\lambda+\rho)\log l(s)}\,|\det L(s)A(s)|^{-1}\,|\Delta(s)|^2 = \Gamma(u).$$

Then it follows that

$$\int_S m_{\lambda,\omega,s}(t)ds = \int_S |\hat{\psi}(L(s)\omega' + \omega(s))|^2 \cdot |\det L(s)A(s)|\,\Gamma(u)ds,$$

where $\omega' = Ad^*(\gamma(t))\omega$. Here we change the variable $s = (u, x)$ to $s' = (u', \xi)$ according to the map $\mathcal{T}_{\omega'} : S \to S$ defined by

$$\begin{cases} u' = u,\\ \xi = L(s)\omega' + \omega(s) = L(s)(\omega' + h(s)).\end{cases}$$

Since

$$\begin{aligned}\frac{\partial\xi}{\partial x_j} &= -L(s)A(s)C_j(u)L(s)(\omega' + h(s)) + L(s)A(s)d_j(u)\\ &= -L(s)A(s)(C_j(u)\xi - d_j(u)),\end{aligned}$$

the Jacobian of $\mathcal{T}_{\omega'}$ is given by

(c) $\det(L(s)A(s))\,\det(C(u) \otimes \xi - D(u))$,

where $C(u) = (C_1(u), \ldots, C_m(u))$ and $D(u) = (d_1(u), \ldots, d_m(u))$. Therefore, if we furthermore assume that

(A7) $\mathcal{T}_{\omega'}$ is of class C^1 and 1:1 outside a set of measure zero,

(A8) $$0 < \iint_{\mathcal{T}_{\omega'}(U\times\boldsymbol{R}^m)} |\hat{\psi}(\xi)|^2\,|\det(C(u) \otimes \xi - D(u))|^{-1}\Gamma(u)d\xi du = c_{\mathfrak{A}} < \infty,$$

then we can deduce that

$$0 < \int_S m_{\lambda,\omega,s}(t)ds = c_{\mathfrak{A}} < \infty.$$

Theorem 2. *If* $\mathfrak{A} = (\lambda, S, l, \Psi)$ *satisfies* (A0)-(A8), *then* $\mathfrak{A}$ *is admissible.*

Remark 3. Let $\mathfrak{A}$ be an admissible quartet in Theorem 2. Since ψ_s is the Dirac function with respect to $t \in \boldsymbol{R}^k$ (see (A2)), Theorem 1 essentially gives an inversion formula for $\mathcal{S}(Q)$. On the other hand, instead of (A1) and (A2) we suppose that

(A1)′ $\mathfrak{q}$ is ideal and $\mathfrak{q} \setminus \bar{\mathfrak{n}}$ is abelian,

(A2)′ $\psi_s(q\gamma(t)) = \delta(q)e^{2\pi i\langle\xi(s),t\rangle}\psi(t)\Delta(s)$
$(q \in Q,\ t \in \boldsymbol{R}^k)$,

where δ is the Dirac function on Q. Then it is easy to see that $\sigma_\omega(\pi_\lambda(l(s))\psi_s)$ is the Fourier multiplier on $L^2(\boldsymbol{R}^k)$ corresponding to $e^{(i\lambda-\rho)\log l(s)}\Delta(s)\mathcal{F}\psi(Ad_0^*(l(s))\xi + \xi(s))$ where $\mathcal{F}\psi$ is the Fourier transform of ψ and Ad_0 is defined by $Ad(l(s)\gamma(t)) = \gamma(Ad_0(l(s))t)$. Therefore, replacing $\boldsymbol{R}^m$ with $\boldsymbol{R}^k$, we can develop the quite same argument on (A3)-(A8) and then, we can deduce an inversion formula for $\mathcal{S}(Q\setminus\bar{N})$. If we combine these two formulas for $\mathcal{S}(Q)$ and $\mathcal{S}(Q\setminus\bar{N})$, we can deduce the one for $\mathcal{S}(\bar{N})$.

4. Examples. We shall give some examples of $L(s)$ and $h(s)$ which satisfy (a) and (b) respectively.

(a1) $L(s)^{-1} = x_1C_1(u) + x_2C_2(u) + \cdots + x_mC_m(u) + C_0(u)$,

where $C_0(u)$ is a (m, m)-matrix. Then (a) is satisfied with $A(s) = I$.

(a2) $L(s)^{-1} = \exp(x_1C_1(u) + x_2C_2(u) + \cdots + x_mC_m(u) + C_0(u))$ and $A(s) = L(s)^{-1}$.

(a3)
$$L(s)^{-1} = \mathrm{diag}(e^{\beta_1(s)}, e^{\beta_2(s)}, \ldots, e^{\beta_m(s)})C_0(u),$$
here $\beta_j(s)$ is the j-th entry of $B(u)x + b_0(u)$ where $B(u) = (b_{ij}(u))$ is a (m, m)-matrix and $b_0(u) \in \boldsymbol{R}^m$. Then (a) is satisfied with $A(s) = \mathrm{diag}(e^{\beta_1(s)}, \ldots, e^{\beta_m(s)})$ and $C_j(u) = \mathrm{diag}(b_{1j}(u), b_{2j}(u), \ldots, b_{mj}(u))C_0(u)$.

(b1) $h(s) = h_0(u)$ and $d_j(u) = 0$,

(b2) $h(s) = L(s)^{-1}b_0(u)$ and
$$d_j(u) = C_j(u)b_0(u),$$

(b3) $h(s) = D(u)x + b_0(u)$ provided $A(s) = I$.

Remark 4. Let U be a subgroup of $GL(m, \boldsymbol{R})$ (see (A3)) and put $\mathscr{D} = |\det(C(u) \otimes \xi - D(u))|$ (see (c)). (1) We define $L(s)$ by (a1) with $C_j(u) = \xi_j uI$ and $C_0(u) = fuI(\xi_j, f \in \boldsymbol{R})$, and $h(s)$ by (b3) with $D(u) = I$ and $b_0(u) = 0$. Then $L(s)^{-1} = (< \Xi, x > + f)u$ $(\Xi = (\xi_1, \xi_2, \ldots, \xi_m))$, $\mathscr{T}_{\omega'}(U \times \boldsymbol{R}^m) = U \times \boldsymbol{R}^m$, and
$$\mathscr{D} = |\det(\Xi \otimes u\xi - I)| = |1 - \langle \Xi, u\xi \rangle|.$$
(2) We suppose that there exists $v \in \boldsymbol{R}^m$ such that $v \otimes D(u) = C(u)$. Then
$$\mathscr{D} = |\det D(u)||\det(v \otimes \xi - I)| = |\det D(u)||1 - \langle v, \xi \rangle|.$$
In this case $v \otimes A(s)D(u) = A(s)C(u)$ and
$$v \otimes \nabla h(s) = \left(\frac{\partial L(s)^{-1}}{\partial x_1}, \ldots, \frac{\partial L(s)^{-1}}{\partial x_m}\right).$$
(3) Let $U = \{e\}$ and $S = \boldsymbol{R}^m$. We define $L(s)$ by (a1) and $h(s)$ by (b3) with $D = I$ and $b_0 = 0$. Then $L(s)^{-1} = C \otimes x + C_0$, $\mathscr{T}_{w'}(\boldsymbol{R}^m) = \xi_{\omega'}(\boldsymbol{R}^m)$ where $\xi_{\omega'}(x) = (C \otimes x + C_0)^{-1}(\omega' + x)$, and $\mathscr{D} = |\det(C \otimes \xi - I)|$.

When $G = SL(n + 2, \boldsymbol{R})$ $(n \geq 1)$ and $\bar{N} \cong H_n$, it is easy to construct the map $l : S \to MA$ for which $L(s)$ is of the above form. Then these examples (1)-(3) yield the inversion formulas (a)-(c) in [4, IV] respectively.

Remark 5. Let $S = \boldsymbol{R}^m$. We define $L(s)$ by (a3) with $C_0 = I$, $B = \mathrm{diag}(a_1, a_2, \ldots, a_m)$ $(a_i \neq 0 \in \boldsymbol{R})$ and $b_0 = 0$, and we let $h(s) = 0$. Then (A4) is satisfied with $A(s) = L(s)^{-1}$ and $C_j = a_jE_{jj}$, (A6) with $\lambda = -\rho$, $\Delta(s) \equiv 1$, and $\Gamma \equiv 1$. Especially, $\mathscr{T}_{\omega'}(\boldsymbol{R}^m) = \Pi_{i=1}^m \mathrm{sgn}(\omega'_i)\boldsymbol{R}_+ = D_{\mathrm{sgn}\omega'}$ and $\mathscr{D} = \Pi_{j=1}^m |a_j\xi_j|$. This is the case treated in [3, §5].

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