Gamma Factors for Generalized Selberg Zeta Functions

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1. Introduction. Let K be an algebraic number field such that $[K:\mathbf{Q}] < \infty$, and $\zeta_K(s)$ be the Dedekind zeta function of K. The completed Dedekind zeta function $\widehat{\zeta_K}(s) = \zeta_K(s) \cdot \Gamma_K(s)$ has the symmetric functional equation: $\widehat{\zeta_K}(1-s) = \widehat{\zeta_K}(s)$. Here, the gamma factor is:

$$\Gamma_K(s) = |D_K|^{\frac{s}{2}} \Gamma_{\mathbf{R}}(s)^{r_1(K)} \Gamma_{\mathbf{C}}(s)^{r_2(K)},$$

where, D_K is the discriminant of K, $r_1(K)$ and $r_2(K)$ are the number of real and complex places of K respectively. We can consider $\Gamma_{\mathbf{R}}(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)$, $\Gamma_{\mathbf{C}}(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1)$ as a "basis" of gamma factors corresponding to infinite places.

In this article we consider **"gamma factors" for Selberg zeta functions.** (cf. Vignéras[6], Sarnak [5], Kurokawa[3]). We give a neat expression of "gamma factors" as in the case of Dedekind zeta functions. (Theorem 1) Furthermore, we obtain a simple proof of the functional equation of the Ruelle zeta function $R(s)$ for a compact $2n$-dimensional real hyperbolic space X (Theorem 2):

$$R(s) \cdot R(-s) = (-4\sin^2(\pi s))^{n \cdot (-1)^{n-1} vol(X)}.$$

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2. Selberg zeta functions. Let G be a connected semisimple Lie group of rank one with finite conter, K be a maximal compact subgroup of G. Let Γ be a co-compact torsion-free discrete subgroup of G. Then $X = \Gamma \backslash G/K$ is a compact locally symmetric space of rank one. For a given irreducible unitary representation τ of K, we denote by $Z_\tau(s)$ the Selberg zeta functions of X with K-type τ as is introduced by Wakayama [7].

For example, let X be a compact Riemann surface of genus $g \geq 2$. Then $X = \Gamma \backslash H$ where $H = SL(2,\mathbf{R})/SO(2)$ is the upper half plane, and Γ is the fundamental group $\pi_1(X)$ discretely embedded in $SL(2,\mathbf{R})$. For trivial τ, the Selberg zeta function $Z(s)$ of a compact Riemann surface is defined by the following Euler products:

$$Z(s) = \prod_{p \in P_\Gamma} \prod_{k=0}^{\infty} (1 - N(p)^{-(k+s)}).$$

Here P_Γ is the set of all primitive hyperbolic conjugacy classes, and the norm function $N(p) = \max\{|\text{eigenvalues of } p|^2\}$. For other rank one Lie groups and non-trivial τ, $Z_\tau(s)$ is defined by similar but more complicated Euler products.

Selberg-Gangolli[2]-Wakayama[7] have shown that:
$Z_\tau(s)$ is meromorphic on $\mathbf{C}$, and tells informations about τ-spectrum:

$$\hat{G}_\tau = \{\pi \in \hat{G} \mid m_\Gamma(\pi) > 0,\ \pi|_K \ni \tau\},$$

where $m_\Gamma(\pi)$ is the multiplicity of a unitary representation π of G in the right regular representation π_Γ of G on $L^2(\Gamma \backslash G)$. (and in our case $m_\Gamma(\pi)$ is finite for all π.)

$Z_\tau(s)$ has moreover the functional equation:

$$(1)\quad Z_\tau(2\rho_0 - s) = \exp\left(\int_0^{s-\rho_0} \Delta_\tau(t)\,dt\right) Z_\tau(s).$$

where, $\rho_0 > 0$ is a constant depending only on G and $\Delta_\tau(t)$ is the "Plancherel" density with K-type τ, whose explicit formula is found in [7]. Hereafter we use **renormalized** ρ_0 and $\Delta_\tau(t)$ like as [4].

3. Gamma factors. we shall express the exponential factor of the functional equation (1) as $\Gamma_\tau(s)/\Gamma_\tau(2\rho_0 - s)$ by the "gamma factor" $\Gamma_\tau(s)$ so that the completed Selberg zeta function $\widehat{Z_\tau}(s) = Z_\tau(s)\Gamma_\tau(s)$ will satisfy the symmetric functional equation:

$$(2)\quad \widehat{Z_\tau}(2\rho_0 - s) = \widehat{Z_\tau}(s)$$

If $\dim X$ is odd, the "Plancherel" density $\Delta_\tau(t)$ is a polynomial and "gamma factor" is trivial. Hereafter we suppose that $\dim X$ is even, i.e. $G = SO(2n, 1), SU(n, 1), Sp(n, 1), F_4$. Then the "Plancherel" density is given by $\Delta_\tau(t) = \sum_{\text{finite sun}} (\text{odd polynomial})\pi(\tan(\pi t))^{\pm 1}$.

Definition 3.1. *We define two "Plancherel polynomials" $P_\tau(t)$ and $Q_\tau(t)$ attached to τ by,*

$$(-1)^{\dim X/2} vol(X)^{-1} \Delta_\tau(t) = -P_\tau(t)\pi\cot(\pi t) + Q_\tau(t)\pi\tan(\pi t).$$

These polynomials are odd polynomials of degree (dim $X - 1$), *whose leading coefficients are positive.*

Theorem 1. (a) *Let $X = \Gamma\backslash G/K$ be an even dimensional compact locally symmetric space of rank one. For $\tau \in \hat{K}$, the gamma factor $\Gamma_\tau(s)$ for $Z_\tau(s)$ is expressed as follows*:

$$\Gamma_\tau(s) = \prod_{l=1}^{\dim X/2} \Gamma_{(0,l)}(s)^{(-1)^{l-1}P_\tau(\dim X/2-l+1)} \times \prod_{l=1}^{\dim X/2} \Gamma_{(1,l)}(s)^{(-1)^{l-1}Q_\tau(\dim X/2-l+\frac{1}{2})}.$$

Here, $\Gamma_{(0,l)}(s)$ and $\Gamma_{(1,l)}(s)$ are "bases" of gamma factors and independent of the representation τ. Let $c(X) = (-1)^{\dim X/2-1}vol(X)$, then these bases are described by the multiple gamma function of order dim X :

$$\Gamma_{(0,l)}(s) = \left[\prod_{k=-(l-1)}^{l-1} \Gamma_{\dim X}\left(s-\rho_0+\frac{\dim X}{2}+k\right)^{(-1)^k\binom{\dim X}{l-|k|-1}}\right]^{c(X)},$$

and

$$\Gamma_{(1,l)}(s) = \left[\prod_{k=0}^{l-1}\left(\Gamma_{\dim X}\left(s-\rho_0+\frac{\dim X}{2}-k-\frac{1}{2}\right)\Gamma_{\dim X}\left(s-\rho_0+\frac{\dim X}{2}+k+\frac{1}{2}\right)\right)^{(-1)^k\binom{\dim X}{l-k-1}}\right]^{c(X)}.$$

(b) *$\Gamma_\tau(s)$ is depending only on "Plancherel polynomials" $P_\tau(t)$ and $Q_\tau(t)$. For $\tau, \tau' \in \hat{K}$,*

$$\Gamma_\tau(s) = \Gamma_{\tau'}(s)$$
$$\Leftrightarrow P_\tau(l) = P_{\tau'}(l) \text{ and } Q_\tau\left(l-\frac{1}{2}\right) = Q_{\tau'}\left(l-\frac{1}{2}\right)$$
$$(l = 1, \ldots, \dim X/2)$$
$$\Leftrightarrow P_\tau(t) = P_{\tau'}(t) \text{ and } Q_\tau(t) = Q_{\tau'}(t)$$
$$\Rightarrow \dim\tau = \dim\tau'$$

Remarks. (1) $\Gamma_r(z)$ is the multiple gamma function as in Kurokawa [3]: $\Gamma_r(z) = \exp\left(\frac{\partial}{\partial s}\zeta_r(s, z)|_{s=0}\right)$, and $\zeta_r(s,z) = \sum_{n_1,\ldots,n_r\geq 0}(n_1+\cdots+n_r+z)^{-s}$ is the multiple Hurwitz zeta function. This normalized multiple gamma function $\Gamma_r(z)$ has many properties similar to the usual gamma function $\Gamma(z)$. For example, $\Gamma_1(z) = (2\pi)^{-\frac{1}{2}}\Gamma(z)$, $\Gamma_0(z) = 1/z$, $\Gamma_r(z+1) = \Gamma_{r-1}(z)^{-1}\cdot\Gamma_r(z)$. etc.

(2) $\Gamma_r(s)$ for trivial τ have been obtained by Kurokawa [4]. Concerning non-trivial τ, only the case $G = SL(2, \mathbf{R})$ has hitherto considered (Sarnak[5]).

(3) "Bases" have a representation-theoretic meaning. Let us consider the case of $G = SO(2n, 1)$.

$$\Gamma_{(1,l)}(s) = \Gamma_{v(l)}(s).$$

$\Gamma_{v(l)}(s)$ is the gamma factor for $Z_{v(l)}(s)$. The representation $v(l) \in \widehat{M}$ satisfies the following:

$$Rep(M) \simeq \mathbf{Z}[v(1), \ldots, v(n)].$$

(See the section of Ruelle zeta functions for notations.) i.e. There is a correspondence between our bases of gamma factors and the basis of $Rep(M)$.

4. Proofs. Let us introduce some polynomials which play key role to prove (a) of Theorem 1.

Proposition 4.1. *For two odd polynomials $P_k(t) = t\prod_{j=1}^{k-1}(t^2-j^2)$ and $Q_k(t) = t\prod_{j=1}^{k-1}\left(t^2-\left(j-\frac{1}{2}\right)^2\right)$, $k \in \mathbf{N}$,*

$$\exp\left(\int_0^{s-\rho_0} P_k(t)\pi\cot(\pi t)dt\right) = \left[\frac{\Gamma_{2k}(\rho_0-s+k)}{\Gamma_{2k}(s-\rho_0+k)}\right]^{-(2k-1)!},$$

and

$$\exp\left(\int_0^{s-\rho_0} Q_k(t)\pi\tan(\pi t)dt\right) = \left[\frac{\Gamma_{2k}(\rho_0-s-\frac{1}{2}+k)\ \Gamma_{2k}(\rho_0-s+\frac{1}{2}+k)}{\Gamma_{2k}(s-\rho_0-\frac{1}{2}+k)\ \Gamma_{2k}(s-\rho_0+\frac{1}{2}+k)}\right]^{\frac{(2k-1)!}{2}}.$$

Proof. Define the multiple sine function $S_r(z) = \Gamma_r(z)^{-1}\Gamma_r(r-z)^{(-1)^r}$, and use the differential equation of $S_r(z)$ [3]:

$$\frac{S_r'}{S_r}(z) = (-1)^{r-1}\binom{z-1}{r-1}\pi\cot(\pi z).$$

□

Next we apply the following lemma, and obtain theorem after some combinatorial calculations.

Lemma 4.2. *For $\tau \in \hat{K}$, $P_\tau(t)$ and $Q_\tau(t)$ are expressed uniquely as $\mathbf{Q}$-linear combination of above polynomials*:

$$P_\tau(t) = \sum_{k=1}^{\dim X/2} a_k(\tau)P_k(t),$$
$$Q_\tau(t) = \sum_{k=1}^{\dim X/2} b_k(\tau)Q_k(t),$$

with $a_k(\tau)$, $b_k(\tau) \in \mathbf{Q}$.

Proof. t^{2i-1} is uniquely expressed by $P_k(t)$'s (resp. $Q_k(t)$'s). For example $t = P_1(t)$, $t^3 = P_2(t) + P_1(t)$. etc. $t = Q_1(t)$, $t^3 = Q_2(t) + \frac{1}{4}Q_1(t)$. etc. And the lemma follows from the fact that $P_\tau(t)$ and $Q_\tau(t)$ are both odd polynomials. □

To prove (b) of the theorem, the following lemma is fundamental:

Lemma 4.3. *Let $f_r(z) = \prod_{k\in\mathbf{Z}}\Gamma_r(z+k)^{a_k}$ for a sequence of rational numbers $\{a_k\}_{k\in\mathbf{Z}}$. Then,*

$$f_r(z) = 1 \Rightarrow \forall a_k = 0.$$

Proof. $f_r(z)/f_r(z+1) = f_{r-1}(z)$ holds by using a property of multiple gamma functions. Therefore, $f_r(z) = 1$ implies $f_{r-1}(z) = 1$. We must prove the case $r = 0$, but this is trivial because $\Gamma_0(z) = 1/z$. □

5. Functional equation of the Ruelle zeta function. Let us consider the case of $G = SO(2n, 1)$. The Ruelle zeta function $R(s)$ of X is defined for $\mathrm{Re}(s) > 2n - 1$ by

$$R(s) = \prod_{p \in P_\Gamma} (1 - N(p)^{-s}),$$

where P_Γ is the set of all primitive hyperbolic conjugacy classes of $\Gamma = \pi_1(X)$ the fundamental group discretely embedded in G, and $N(p)$ is the norm function. Fried [1] shows that $R(s)$ can be written as a product of generalized Selberg zeta functions:

$$R(s) = \prod_{l=1}^{2n} Z_{v(l)}(s + l - 1)^{(-1)^{l-1}},$$

$v(l) : M \to \wedge^{l-1}(\mathbf{C}^{2n-1})$ standard representations. where M is the centralizer of A in K under the Iwasawa decomposition $G = KAN$. In our case, $K = SO(2n)$ and $M = SO(2n-1)$. We know that the gamma factor of $Z_{v(l)}(s)$ is $\Gamma_{v(l)}(s) = \Gamma_{(1,l)}(s)$ from Theorem 1, $\rho_0 = n - \dfrac{1}{2}$ and $\dim X = 2n$:

$$\Gamma_{v(l)}(s) = \left[\prod_{k=0}^{l-1} (\Gamma_{2n}(s-k)\Gamma_{2n}(s+k+1))^{(-1)^k\binom{2n}{l-k-1}}\right]^{c(X)}.$$

Theorem 2. *Let $X = \Gamma \backslash SO(2n, 1)/SO(2n)$ be a compact real hyperbolic space. Then the Ruelle zeta function $R(z)$ of X has the following functional equation:*

(3) $$R(z) \cdot R(-z) = (-4\sin^2(\pi z))^{n \cdot c(X)}.$$

Here, $c(X) = (-1)^{n-1} vol(X)$.

Proof. $R(z) \cdot R(-z)$

$$= \prod_{l=1}^{n} \left[\frac{Z_{v(l)}(z+l-1)}{Z_{v(l)}(-z+2n-l)} \frac{Z_{v(l)}(-z+l-1)}{Z_{v(l)}(z+2n-l)}\right]^{(-1)^{l-1}}$$

$$= \prod_{l=1}^{n} \left[\prod_{k=0}^{l-1} (S_{2n}(z+l-k-1) S_{2n}(z+l+k))^{(-1)^k\binom{2n}{l-k-1}}\right]^{(-1)^{l-1} \cdot c(X)}$$

$$\times \prod_{l=1}^{n} \left[\prod_{k=0}^{l-1} (S_{2n}(-z+l-k-1) S_{2n}(-z+l+k))^{(-1)^k\binom{2n}{l-k-1}}\right]^{(-1)^{l-1} \cdot c(X)}$$

$$= \prod_{j=0}^{2n-1} (S_{2n}(z+j) S_{2n}(-z+j))^{a(j) \cdot c(X)}$$

$$a(j) = \begin{cases} (-1)^j(n-j)\binom{2n}{j} + (-1)^{j-1}b(j) & \cdots j = 0, \ldots, n \\ (-1)^{j-1}b(j) & \cdots j = n+1, \ldots, 2n-1 \end{cases}$$

$$b(j) = \sum_{\frac{j+1}{2} \le l \le \min(n,j)} \binom{2n}{2l-j-1} = b(2n-j)$$

$$= \prod_{j=0}^{n-1} (S_{2n}(z+j) S_{2n}(-z+j))^{(a(j)-a(2n-j)) \cdot c(X)}$$

$$= \prod_{j=0}^{n-1} (S_{2n}(z+j) S_{2n}(-z+j))^{(-1)^j(n-j)\binom{2n}{j} \cdot c(X)}$$

$$= (S_1(s) \cdot S_1(-z))^{n \cdot c(X)}$$

$$= (-4\sin^2(\pi z))^{n \cdot c(X)}.$$ Q.E.D. □

Remarks. We have used the known properties of multiple sine functions such as following in above calculations.

$$S_1(z) = \Gamma_1(z)^{-1}\Gamma_1(1-z)^{-1}$$
$$= [(2\pi)^{-\frac{1}{2}}\Gamma(z) \cdot (2\pi)^{-\frac{1}{2}}\Gamma(1-z)]^{-1}$$
$$= 2\sin(\pi z).$$

and

$$S_{2n}(\alpha) \cdot S_{2n}(2n - \alpha) = 1,$$

and

$$S_1(z) \cdot S_1(-z)$$
$$= \prod_{j=0}^{2n-1} (S_{2n}(z+j) S_{2n}(-z+j))^{(-1)^j\binom{2n-1}{j}}$$
$$= \prod_{j=0}^{n-1} (S_{2n}(z+j) S_{2n}(-z+j))^{\left\{(-1)^j\binom{2n-1}{j} - (-1)^{2n-j}\binom{2n-1}{2n-j}\right\}}$$
$$= \prod_{j=0}^{n-1} (S_{2n}(z+j) S_{2n}(-z+j))^{(-1)^j\frac{(n-j)}{n}\binom{2n}{j}}.$$

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