On Cartier-Voros Type Selberg Trace Formula for Congruence Subgroups of $PSL(2, \mathbf{R})$

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(Communicated by Shokichi IYANAGA, M. J. A., Sept. 12, 1995)

1. Introduction. Let Γ be a discrete subgroup of $G = PSL(2, \mathbf{R})$. The group Γ acts on the upper half plane $\mathbf{H}$ by the usual linear fractional transformation. We assume that the fundamental domain of Γ, which is denoted by $\Gamma\backslash\mathbf{H}$, is a finite volume surface with the hyperbolic metric. The Laplacian Δ acting on the space $L^2(\Gamma\backslash\mathbf{H})$ has the spectrum consisting of the discrete and continuous spectra in general. In this setting, as two different expression of the trace of an G-invariant integral operator $L : f(z) \to \int_{\Gamma\backslash\mathbf{H}} \hat{K}(z, z')f(z')dz' (f \in L^2(\Gamma\backslash\mathbf{H}))$ with the kernel function

$$\hat{K}(z, z') = \sum_{\sigma\in\Gamma} k(\sigma z, z') - \hat{H}(z, z'),$$

where k is a point pair invariant and $\hat{H}(z, z')$ is so defined that the continuous spectrum of Δ disappears, Selberg showed his famous trace formula of the following form (see [9]): for any function $h(\rho)$ $(\rho \in \mathbf{C})$, which we call a test function, satisfying the condition A below,

$$D(h) = I(h) + H(h) + E(h) + CP(h),$$

where the left hand side $D(h) = \sum_{n=0}^{\infty} h(\rho_n)$ is the expansion of $Tr(L) = \int_{\Gamma\backslash\mathbf{H}} \hat{K}(z, z)dz$ as the sum of the eigenvalue $h(\rho_n)$ of L corresponding to the discrete spectrum $\lambda_n = \frac{1}{4} + \rho_n^2$ of Δ, i.e. $L\varphi_n = h(\rho_n)\varphi_n$ for $\Delta\varphi_n = \lambda_n\varphi_n$, the right hand side is the expansion of $Tr(L)$ with respect to the conjugacy classes of Γ, and $I(h)$ (resp. $H(h)$, $E(h)$) is the contribution of the identity (resp. hyperbolic, elliptic) conjugacy class of Γ, and $CP(h)$ is the sum of the contribution of the parabolic conjugacy classes of Γ and the contribution of $\hat{H}(z, z')$. This formula has been one of the important objects of study in analytic number theory. Especially the studies of the relations among the Selberg zeta functions $Z_\Gamma(s)$ (see §2) which is induced from the term $H(h)$ of this formula for a special test function, the arithmetic zeta functions, and the spectral zeta functions are very interesting in view of the 'unifying' theory of various zeta functions.

The condition A for a test function $h(\rho)$ is as follows:

(1) $h(-\rho) = h(\rho)$,

(2) $h(\rho)$ is holomorphic in the strip $|\mathrm{Im}\,\rho| < c \left(\text{for some } c > \frac{1}{2}\right)$,

(3) $h(\rho) = O(|\rho|^{-\alpha})$ $(\alpha > 2)$ as $|\rho| \to \infty$ in the above strip.

According to the condition A, for example, $h(t, \rho) = e^{-t\rho} (t > 0)$ can not be taken as a test function.

Now we consider the theta type function $\Theta_\Gamma(t) = \sum_{n=0}^{\infty} e^{-t\rho_n} (t > 0)$ associated with the operator $\sqrt{\Delta - \frac{1}{4}}$ on $L^2(\Gamma\backslash\mathbf{H})$ (see §4). This function $\Theta_\Gamma(t)$ is very interesting because of the following view points: first, $\Theta_\Gamma(t)$ is an analogue of theta functions associated with Δ (see [8]); and secondly, $\Theta_\Gamma(t)$ is also an analogue of that associated with the zeros of zeta functions (see [1], [4], [5]). However, as stated above, the Selberg trace formula can be of no help to study $\Theta_\Gamma(t)$.

But for co-compact discrete subgroups Γ, Cartier-Voros[2] showed the modified Selberg trace formula to study $\Theta_\Gamma(t)$. In this case, Γ has no elliptic and parabolic conjugacy class and no continuous spectrum. This Cartier-Voros type modified formula demands the following condition B for a test function $h(\rho)$.

(1) $h(\rho)$ is holomorphic in an open set $V \subset \mathbf{C}$ containing the closed half plane $\mathrm{Re}\,\rho \geq 0$,

(2) $h(\rho) = O(|\rho|^{-\alpha})$ $(\alpha > 2)$ as $|\rho| \to \infty$ in the strip $|\mathrm{Im}\,\rho| < c$(for some $c > 0$),

(3) $h(\rho)d\log Z_\Gamma\left(\frac{1}{2} + i\rho\right)$ is integrable in $-\frac{\pi}{2} \leq \arg\rho \leq 0$, and $h(\rho)d\log Z_\Gamma\left(\frac{1}{2} - i\rho\right)$ is integrable in $0 \leq \arg\rho \leq \frac{\pi}{2}$.

Here 'integrable in the sector $\mathbf{S}$' means that for any two contours C, C' in $\mathbf{S}$ joining a some $\rho_0 \in \mathbf{S}$ to ∞, the integral along C equals that along C'. So one can apply the Cartier-Voros type Selberg trace formula for $h(t, \rho)$ and can study the properties of $\Theta_\Gamma(t)$.

From our arithmetic interests, we wish to discuss when Γ is a congruence subgroup, which has a noncompact fundamental domain. In the following sections, we extend the Cartier-Voros type Selberg trace formula for such Γ (§3), and give the properties of $\Theta_\Gamma(t)$ as its applications (§4). Note that our $\Theta_\Gamma(t)$ is many-valued. This fact does not occur in the case of co-compact Γ (see [2]; also compare with [1], [5]).

I should like to express my gratitude to Professor N. Kurokawa for his valuable guidance, and to Professer S. Koyama for useful discussions.

2. Determinant expression. In this section, we review the determinant expression of Selberg zeta functions that we need in §3. The details can be found in [6].

Let Γ be the image of one of the typical congruence subgroups $\Gamma_i(N) \subset SL(2, \mathbf{Z})\ (i = 0,1,2)$ by the natural quotient map $SL(2, \mathbf{R}) \to PSL(2, \mathbf{R})$. Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the eigenvalues of the Laplace operator Δ on $L^2(\Gamma\backslash\mathbf{H})$. Then the discrete part of the determinant of $\Delta - s(1-s)$ is defined by

$$\det{}_D(\Delta - s(1-s)) := \prod_{n=0}^{\infty}{}'(\lambda_n - s(1-s)) \quad \text{for } s > 1,$$

where Π' means a regularized product (see [5], [6], [7]). Now we denote the Selberg zeta function by

$$Z_\Gamma(s) := \prod_P \prod_{k=0}^{\infty} (1 - N(P)^{-(s+k)}) \quad \text{for } \mathrm{Re}\, s > 1,$$

where P runs through the primitive hyperbolic conjugacy classes of Γ, and $N(P)$ is its norm. Then one can prove that $Z_\Gamma(s)$ has a meromorphic continuation to the whole s-plane.

Then there exists a relation between $\det_D(\Delta - s(1-s))$ and $Z_\Gamma(s)$ called determinant expression of Selberg zeta functions, that is (see [6])

$$\det{}_D(\Delta - s(1-s))\det{}_C(\Delta, s)e^{c+c's(1-s)} = Z_I(s)Z_\Gamma(s)Z_E(s)Z_P(s)$$

where $\det_C(\Delta, s)$ is the contribution of the continuous spectrum of Δ, and $Z_I(s)$ (resp. $Z_E(s)$, $Z_P(s)$) is the contribution of the identity (resp. elliptic, parabolic) conjugacy class of Γ, and c, c' are certain constants depending on Γ. From this identity, $\det_D(\Delta - s(1-s))$ is defined for all $s \in \mathbf{C}$ except for the poles of the right hand side. This relation contains the informations about the zeros and the poles of $Z_P(s)$ and gives the symmetric functional equation. That is, if we put $\hat{Z}(s) := Z_I(s)Z_\Gamma(s)Z_E(s)Z_{CP}(s)$, $Z_{CP}(s) := Z_P(s)\det_C^{-1}(\Delta, s)$ then $\hat{Z}(s) = \hat{Z}(1-s)$ holds. The determinant expression of $Z_\Gamma(s)$ plays a key role in the next section.

3. Cartier-Voros type Selberg trace formula. First we introduce the following notations. We consider an operator $\sqrt{\Delta - \frac{1}{4}}$ on $L^2(\Gamma\backslash\mathbf{H})$, and normalize its eigenvalue as

$$\rho_n = \begin{cases} \left(\lambda_n - \frac{1}{4}\right)^{\frac{1}{2}}, & \text{if } \lambda_n \geq \frac{1}{4}, \\ -i\left(\frac{1}{4} - \lambda_n\right)^{\frac{1}{2}}, & \text{if } 0 \leq \lambda_n < \frac{1}{4}. \end{cases}$$

We denote the number of the classes of order 2(resp. 3) by n_2 (resp. n_3), and the number of inequivalent cusps by K. For the scattering matrix $\Phi(s)$ of Γ whose entries come from the constant term of the Eisenstein series, we define the constant K_0 as $-\lim_{s\to\frac{1}{2}}\Phi(s)$. We denote by $\Lambda(n)$ $(n \in \mathbf{N})$ the von Mangoldt function. Then our first main theorem can be stated as follows.

Theorem 1. *Let $h(\rho)$ be a function of a complex variable ρ which satisfies the condition B* (see §1). *Moreover when Δ has the eigenvalue $\frac{1}{4}$, we assume that $h(0) = 0$. And let $g(u)$ be a Fourier transform of $h(\rho)$:*

$$g(u) := \frac{1}{2\pi}\int_0^\infty (e^{i\rho u} + e^{-i\rho u})h(\rho)\,d\rho.$$

Then the following identity holds:

(1) $\mathscr{D}(h) = \mathscr{I}_0(h) + \mathscr{H}(h) + \mathscr{E}(h) + \mathscr{CP}(h)$,

where

$$\mathscr{D}(h) = \sum_{n=0}^{\infty} h(\rho_n)$$

$$\mathscr{I}_0(h) = \frac{vol(\Gamma\backslash\mathbf{H})}{2\pi}\int_0^\infty h(\rho)\rho\tanh \pi\rho\, d\rho$$

$$\mathscr{H}(h) = \int_0^\infty \frac{h(-i\kappa) - h(i\kappa)}{2\pi i}\, d\log Z_\Gamma\left(\frac{1}{2} + \kappa\right)$$

$$\mathscr{E}(h) = \frac{n_2}{2}\int_0^\infty \frac{1}{e^{\rho\pi} + e^{-\rho\pi}}\, h(\rho)\, d\rho + \frac{2n_3}{3\sqrt{3}}\int_0^\infty \frac{e^{\frac{1}{3}\rho\pi} + e^{-\frac{1}{3}\rho\pi}}{e^{\rho\pi} + e^{-\rho\pi}}\, h(\rho)\, d\rho$$

$$\mathscr{CP}(h) = -Kg(0)\log 2 + \frac{K+K_0}{4}h(0)$$
$$-\frac{K}{2\pi}\int_0^\infty \left\{\frac{\Gamma'}{\Gamma}(1+i\rho) + \frac{\Gamma'}{\Gamma}(1-i\rho)\right\}h(\rho)d\rho$$
$$-\frac{K}{2\pi}\int_0^\infty \left\{\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+i\rho\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}-i\rho\right)\right\}h(\rho)d\rho$$
$$+ g(0)\log\frac{\pi^K}{A} + 2\sum_\chi\sum_{n=1}^\infty \frac{\Lambda(n)\chi(n)}{n}g(2\log n),$$

the contour in $\mathscr{H}(h)$ *avoids the poles* $\kappa_n\left(0<\kappa_n \le \frac{1}{2}\right)$ *to the right, A is a positive integer composed of the primes dividing N, and the product over* χ *has K terms, in each of which* χ *is a Dirichlet character to some modulus dividing N (cf. [6]).*

Proof. We consider the following Cauchy integral

$$\mathscr{Q}(h) = \frac{1}{2\pi i}\int_C h(-i\kappa)d\log \det{}_D\left(\Delta - \frac{1}{4}+\kappa^2\right)$$

where C is a suitable contour containing all $i\rho_n$ in the inside. We calculate $\mathscr{Q}(h)$ by two different ways.

First, according to the definition of $\det_D$, a little computation shows

$$\mathscr{Q}(h) = \frac{1}{2\pi i}\sum_{n=0}^\infty \int_C h(-i\kappa)\cdot\frac{2\kappa}{\rho_n^2+\kappa^2}d\kappa$$
$$= \sum_{n=0}^\infty h(\rho_n),$$

where the last equality follows from the residue theorem. In this computation, we remark the fact that the series $\sum_{n=0}^\infty \rho_n^{-2-\delta}$ is convergent for $\delta > 0$.

Secondly we divide the contour C into two parts corresponding to $\mathrm{Re}\,\kappa > 0$ and $\mathrm{Re}\,\kappa < 0$, and denote by C^+, C^- the respective parts. From the determinant expression in §2, $\det_D\left(\Delta - \frac{1}{4} + \kappa^2\right)$ is equal to

$$e^{-c-c'(\frac{1}{4}-\kappa^2)}Z_I\left(\frac{1}{2}+\kappa\right)Z_\Gamma\left(\frac{1}{2}+\kappa\right)$$
$$Z_E\left(\frac{1}{2}+\kappa\right)Z_{CP}\left(\frac{1}{2}+\kappa\right), \text{ on } C^+$$
$$e^{-c-c'(\frac{1}{4}-\kappa^2)}Z_I\left(\frac{1}{2}-\kappa\right)Z_\Gamma\left(\frac{1}{2}-\kappa\right)$$
$$Z_E\left(\frac{1}{2}-\kappa\right)Z_{CP}\left(\frac{1}{2}-\kappa\right), \text{ on } C^-.$$

Since each of $Z_X(X = I, E, CP)$ for our Γ are known by Koyama[6], we can compute explicitly the integrals

$$\int_{C^+} h(-i\kappa)d\log Z_X\left(\frac{1}{2}+\kappa\right)$$
$$+\int_{C^-} h(-i\kappa)d\log Z_X\left(\frac{1}{2}-\kappa\right).$$

This yields the right hand side of the identity (1), for the Stirling formula and the assertion (2), (3) of the condition B allow us to change the contour $C^+(C^-)$ with the one along the real or imaginary axis. Hence the proof is complete. □

4. Application of theorem 1. In this section, we assume that Δ has no eigenvalues equal to $\frac{1}{4}$, and we consider the theta type function $\Theta_\Gamma(t)$ associated with the operator $\sqrt{\Delta - \frac{1}{4}}$ which is defined by

$$\Theta_\Gamma(t) := \sum_{n=0}^\infty e^{-t\rho_n}, \quad \text{for } t > 0.$$

Now we fix $t > 0$, and we put $h(t, \rho) := e^{-t\rho}$. Since $h(t,\rho)$ as a function of ρ satisfies the assumptions of theorem 1, we can apply theorem 1 to this function. Then the left-hand side $\mathfrak{D}(h)$ of (1) becomes $\Theta_\Gamma(t)$. Thus we obtain our second main theorem.

Theorem 2. *The theta function* $\Theta_\Gamma(t)$ *has the following properties.*

(1) $\Theta_\Gamma(t)$ *has the meromorphic continuation to the region* $\mathfrak{D} := \mathbf{C} - \{x \in \mathbf{R} : x \le 0\}$.

(2) $\Theta_\Gamma(t)$, $t \in \mathfrak{D}$ *satisfies the functional equation.*

$$\Theta_\Gamma(t) + \Theta_\Gamma(-t) = \frac{vol(\Gamma\backslash\mathbf{H})}{4\pi}\cdot\frac{\cos\frac{t}{2}}{\sin^2\frac{t}{2}}$$
$$+\frac{n_2}{4}\sec\frac{t}{2} + \frac{n_3}{3\sqrt{3}}\left\{\csc\left(\frac{t}{2}+\frac{\pi}{3}\right)+\csc\left(\frac{t}{2}+\frac{2}{3}\pi\right)\right\}$$
$$+\frac{K+K_0}{2} - \frac{K}{2}\left(1 \mp i\cot\frac{t}{4}\right),$$

where the double sign depends on whether $\mathrm{Im}\,t > 0$ *or* $\mathrm{Im}\,t < 0$.

(3) $\Theta_\Gamma(t)$, $t \in \mathfrak{D}$ *has at most simple poles at*
$\pm\, im\log N(P)$
(P : *prim. hyp. conj. class,* $m = 1,2,3,\ldots$),
$\pm\, 2im\log p$ (p : *prime number,* $m = 1,2,3,\ldots$).

Proof. By a little calculation of the right side of the identity (1) for $h(t,\rho)$, we have the following expression of $\Theta_\Gamma(t)$ for $t > 0$:

$$\Theta_\Gamma(t) = \frac{vol(\Gamma\backslash\mathbf{H})}{2\pi}\left\{\frac{1}{t^2} + 2\sum_{m=1}^\infty \frac{(-1)^m}{(t+2\pi m)^2}\right\}$$
$$+\frac{1}{\pi}\int_0^a \sin\kappa t d\log Z_\Gamma\left(\frac{1}{2}+\kappa\right)$$

$$+\frac{1}{2\pi}\sum_{P}\sum_{m=1}^{\infty}\frac{K_{P,m}}{N(P)^{ma}}\left(\frac{e^{iat}}{t+im\log N(P)}+\frac{e^{-iat}}{t-im\log N(P)}\right)$$
$$+\frac{n_2}{2}\sum_{m=0}^{\infty}\frac{(-1)^m}{t+(2m+1)\pi}$$
$$+\frac{2n_3}{3\sqrt{3}}\left(\sum_{m=0}^{\infty}\frac{(-1)^m}{t+\left(2m+\frac{2}{3}\right)\pi}+\sum_{m=0}^{\infty}\frac{(-1)^m}{t+\left(2m+\frac{4}{3}\right)\pi}\right)$$
$$+\frac{1}{\pi t}\log\frac{\pi^K}{2^K A}+\frac{K+K_0}{4}-\frac{K}{2\pi}\mathscr{CP}_{3,4}(t)$$
$$+\frac{1}{\pi}\sum_{\chi}\sum_{n=1}^{\infty}\frac{\Lambda(n)\chi(n)}{n}\left(\frac{1}{t-2i\log n}+\frac{1}{t+2i\log n}\right),$$

where $a>\frac{1}{2}$ is any constant and

$$K_{P,m}=\frac{\log N(P)}{2\sinh\frac{m\log N(P)}{2}},$$

$$\mathscr{CP}_{3,4}(t)=2\int_0^{\infty}\left\{\frac{\Gamma'}{\Gamma}(1+2i\rho)+\frac{\Gamma'}{\Gamma}(1-2i\rho)\right\}e^{-t\rho}d\rho-\frac{4\log 2}{t}.$$

Clearly each term of the right hand side of the above identity except $\mathscr{CP}_{3,4}(t)$ has a meromorphic continuation to the region $t\in\mathfrak{D}$, and its poles are simple and at $\pm im\log N(P)$, $\pm 2im\log p$. Using the formula $\Gamma(1+z)\Gamma(1-z)=\pi z\csc \pi z$, we have

$$\frac{\Gamma'}{\Gamma}(1+2i\rho)+\frac{\Gamma'}{\Gamma}(1-2i\rho)=2\frac{\Gamma'}{\Gamma}(1+2i\rho)+i\left(\frac{1}{2\rho}-\pi\coth 2\pi\rho\right).$$

Hence we have the expression

$$\mathscr{CP}_{3,4}(t)=\frac{4}{t}\int_0^{\infty}\frac{\Gamma'}{\Gamma}\left(1+\frac{2i\rho}{t}\right)e^{-\rho}d\rho+i\left\{\frac{\Gamma'}{\Gamma}\left(\frac{t}{4\pi}\right)+\frac{2\pi}{t}-\log\frac{t}{4\pi}\right\}-\frac{4\log 2}{t},$$

for $t>0$. Therefore $\mathscr{CP}_{3,4}(t)$ has the analytic continuation to $-\pi<\arg t<\frac{\pi}{2}$. Similarly from the expression

$$\mathscr{CP}_{3,4}(t)=\frac{4}{t}\int_0^{\infty}\frac{\Gamma'}{\Gamma}\left(1-\frac{2i\rho}{t}\right)e^{-\rho}d\rho-i\left\{\frac{\Gamma'}{\Gamma}\left(\frac{t}{4\pi}\right)+\frac{2\pi}{t}-\log\frac{t}{4\pi}\right\}-\frac{4\log 2}{t},$$

$\mathscr{CP}_{3,4}(t)$ has the analytic continuation to $-\frac{\pi}{2}<\arg t<\pi$. Hence the first assertion (1) of the theorem is proved. Now (2) and (3) are clear from the explicit expression of $\Theta_\Gamma(t)$. The proof is complete. □

Remark 1. When Δ has the eigenvalue $\lambda_n=\frac{1}{4}$, Theorem 2 remains true if we suppose that the contribution of the term $e^{-t\rho_n}$ of $\Theta_\Gamma(t)$ for $\rho_n=0$ equals to $\frac{1}{2}$.

Remark 2. According to the expression of $\Theta_\Gamma(t)$, it is a many-valued function of $t\in\mathbf{C}$. This property is not observed when Γ is a co-compact subgroup.

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