Extremal Kähler Metrics and the Calabi Energy*)

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(Communicated by Kiyosi ITÔ, M. J. A., June 13, 1995)

Let (M, Ω) be a compact complex manifold with a distinguished Kähler class. By abuse of terminology, we say a Kähler metric g is in Ω if the Kähler form ω of g is in Ω. The volume and total scalar curvature of a Kähler metric in Ω depend only on Ω, and are denoted V_Ω and S_Ω respectively. The *Calabi energy* of a metric is defined to be

$$\Phi_\Omega(g) = \int_M s_g^2 \, \mathrm{dvol}_g, \tag{1}$$

and is obviously bounded below by S_Ω^2/V_Ω. A critical metric for the Calabi energy-among metrics representing the same Kähler class-is called an *extremal* Kähler metric.

Calabi ([3], page 99) asked whether or not the functional Φ_Ω has a unique critical value, and if so, whether or not extremal metrics are in fact global minima of the energy in their Kähler class. The main result of this note is to announce that the answer to both questions is "Yes." Moreover, the critical energy can be determined *a priori,* without reference to an extremal metric. (This was proven independently by Simanca [8].) The proof relies on three ingredients: The Euler equation for critical metrics of the Calabi energy; the Futaki character; and a natural complex-bilinear pairing, due to Futaki and Mabuchi [4], on the space of holomorphic gradient vector fields. We state the precise result, outline the proof, then suggest some possible consequences.

Let $\mathcal{F}_\Omega$ denote the Futaki character of Ω, and let X_Ω denote an extremal Kähler vector field, see Futaki and Mabuchi [4]. Intuitively, the vector field X_Ω is dual to the character $\mathcal{F}_\Omega$ under a canonical complex-bilinear pairing on the space of holomorphic gradient vector fields, but because this pairing is degenerate, the vector field X_Ω is not well-defined. However, the value $\mathcal{F}_\Omega(X_\Omega)$ *is* well-defined (see [4]), and is a nonnegative real number (see [5]).

*) The author was supported by JSPS Fellowship #P-94016.

Theorem A. *For each metric g with fundamental class Ω,*

$$\Phi_\Omega(g) \geq S_\Omega^2/V_\Omega + \mathcal{F}_\Omega(X_\Omega), \tag{2}$$

with equality if and only if g is a critical metric for Φ_Ω.

This result highlights the close relationship between the Futaki character and the Calabi energy, and has an amusing statistical interpretation. The quantity $\Phi_\Omega(g) - S_\Omega^2/V_\Omega$ is the variance of the scalar curvature function s_g, computed on the measure space (M, dvol_g). Theorem A says that this variance is *a priori* bounded below by $\mathcal{F}_\Omega(X_\Omega)$ as the metric g ranges in Ω. In other words, equation (2) asserts that the norm squared of the Futaki character is a precise numerical measure of how far an extremal metric is from having constant scalar curvature. This nicely encodes both the result of Futaki that a Kähler class containing a metric with constant scalar curvature must have vanishing Futaki character, and the result of Calabi that if the Futaki character $\mathcal{F}_\Omega$ vanishes, then an extremal metric in Ω has constant scalar curvature.

The idea of the proof is quite simple. Fix any Kähler metric g whose Kähler form ω represents the class Ω. There is a finite-dimensional space of smooth, complex-valued functions, called *g-holomorphy potentials,* whose gradients with respect to g are holomorphic. Write the scalar curvature function s of g as an L_g^2-orthogonal sum

$$s = \mathbf{H}s + \Pi s + s^\perp, \tag{3}$$

where $\mathbf{H}s = S_\Omega/V_\Omega$ is the average scalar curvature (i.e. the g-harmonic part), and $\mathbf{H}s + \Pi s$ is the g-orthogonal projection of s into the space of g-holomorphy potentials. (*n.b.* The projection Π here is written $\Pi_g - \mathbf{H}$ in [5].) Since M is compact, the metric g is extremal if and only if the gradient field of the scalar curvature is holomorphic (see for example, [2]). This is the same as saying the scalar curvature s is a holomorphy potential, or that $s^\perp = 0$.

Now, the L_g^2-norm of $\mathbf{H}s$ is S_Ω^2/V_Ω, and the L_g^2-norm of $s^\perp$ is non-negative, and equal to zero if and only if g is extremal. Theorem A follows immediately from the fact that the L_g^2-norm of the function Πs is equal to $\mathscr{F}_\Omega(X_\Omega)$, and in particular is independent of g. The proof of this fact is a trifle delicate because the function Πs is not necessarily real-valued, and will appear elsewhere [5]. (I would like to thank Santiago Simanca for very helpful discussions on this matter.)

The Euler equation for Φ_Ω makes it seem quite unlikely that there is a further "universal" decomposition of $s^\perp$ into terms whose L_g^2-norm is independent of g. If there were a non-trivial projector Π_1 depending on g, and such that the norm of $\Pi_1 s^\perp$ were independent of g, one would expect to find metrics for which $s^\perp = \Pi_1 s^\perp \neq 0$. Such a metric would be a critical metric with energy strictly larger than that allowed by Theorem A. This lends *heuristic* evidence to an affirmative answer of the following:

Problem. *Is it true that for every* (M, Ω),

$$\inf_g \Phi_\Omega(g) - (S_\Omega^2/V_\Omega + \mathscr{F}_\Omega(X_\Omega)) = 0,$$

where the infimum is taken over all metrics with fundamental class Ω?

If the answer turns out to be "no," then a positive bound of the left-hand side—expressed in terms of *a priori* data—would yield a new obstruction to existence of an extremal metric.

There are non-trivial obstructions to existence of an extremal metric due to Levine [7], Calabi [3], and Mabuchi and the author [6], which depend on the structure of the Lie algebra of holomorphic vector fields. These obstructions are vacuous unless the nilpotent radical (of the Lie algebra of holomorphic vector fields) is non-trivial. Such obstructions cannot manifest themselves as positive contributions to the Calabi energy because holomorphy potentials of vector fields in the nilpotent radical are always g-orthogonal to Πs and $s^\perp$.

There are also examples of Kählerian surfaces, due to Burns and de Bartolomeis [1], which have *no* holomorphic vector fields and yet do not admit an extremal metric. A general explanation of this behavior does not exist at present, but the author believes it quite unlikely that this phenomenon is manifested by the Calabi energy. It seems much more likely that if one takes a minimizing sequence of metrics, their energies approach $S_\Omega^2/V_\Omega = 0$, while the condition of being Kähler with respect to the "pathological" complex structure causes the metrics to diverge in an essential way.

References

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