A Three-manifold Invariant Derived from the Universal Vassiliev-Kontsevich Invariant[†)]

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We construct a three manifold invariant from the universal Vassiliev-Kontsevich (UVK) invariant $\hat{Z}$, which includes Lescop's generalization of the Casson-Walker invariant.

The UVK invariant $\hat{Z}$ has values in the space $\mathscr{A}$ of chord diagrams subject to the four term relation [2, 1, 3]. Here, we construct a three manifold invariant by taking Kirby move (Fig. 1) invariant part from the UVK invariant of framed links $\hat{Z}_f$ constructed in [3]. We modify $\hat{Z}_f$ so that it has a good property with respect to the KII moves. Let $\nu = Z_f(U)^{-1}$, which is the factor introduced in [3] to normalize the effect of maximal and minimal points. For an l-component link L, let $\check{Z}_f(L) = \hat{Z}_f(L) \# (\nu, \nu, \cdots, \nu)$. This means that we connect-sum ν to each string of $\hat{Z}_f(L)$. Then, we take a certain quotient $\bar{\mathscr{A}}$ of $\mathscr{A}$ so that the image of $\check{Z}_f(L)$ is stable under the KII moves. Let $\Lambda'(L)$ denote this image of $\check{Z}_f(L)$, then $\Lambda'(L)$ factors the Jones-Witten invariant in [6, 9] except the normalization factor for the KI moves. We study a low degree part of $\Lambda'(L)$ concretely, and, after normalizing for the KI moves, we show that $\Lambda'(L)$ includes the order of the first homology and Casson's invariant. For a $\mathbf{Z}/r\mathbf{Z}$-homology 3-sphere (r: odd prime), the Jones-Witten invariant dominates the Casson-Walker invariant [7].

In this short note, we expose the results and the idea of proofs. The detail will be given elsewhere.

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1. Modified universal Vassiliev-Kontsevich invariant. We use notations in [3, 4]. Let C be a chord diagram with a distinguished string s, and let k be the number of end point of chords on s. Let $\Delta(C)$ denote the sum of 2^k diagrams obtained by adding a string parallel to s and changing each point on s as in Fig. 2.

Proposition 1. *Let L and L' be two links as in the KII move in Fig.* 1. *Then $\check{Z}_f(L')$ is obtained from $\check{Z}_f(L)$ as in Fig.* 3.

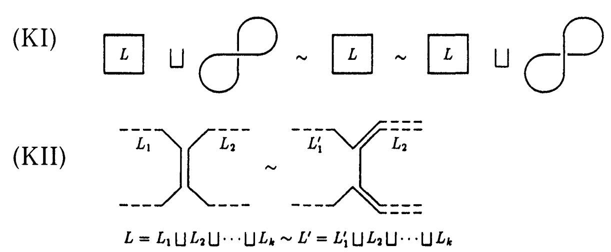

Fig. 1. Kirby Moves

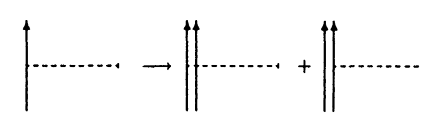

Fig. 2. Parallel of a chord diagram.

To prove this proposition, we use the result in [4].

Let $\mathscr{A}^{(l)}$ denote the $\mathbf{C}$-linear space spanned by the chord diagrams on a disjoint union of l S^1's subject to the four term relation. We add two types of relations to $\mathscr{A}^{(l)}$. The first one is for orientations of strings. Let D be a chord diagram and let D' be a chord diagram obtained by changing the orientation of a string s of D. Then we impose $D' \sim (-1)^{e(s)} D$, where $e(s)$ denote the number of end points on s. We call this the orientation independence relation. The second one is for the KII move given in Fig. 4. We call it the KII relation of chord diagrams. Let $\bar{\mathscr{A}}^{(l)} = \mathscr{A}^{(l)}/(\text{Orientation independence relation, KII relation})$, and let $\Lambda'(L)$ be the image of $\check{Z}_f(L)$ in $\bar{\mathscr{A}}^{(l)}$

for an l-component link L.

Proposition 1′. *$\Lambda'(L)$ is invariant under KII moves and orientation change of any*

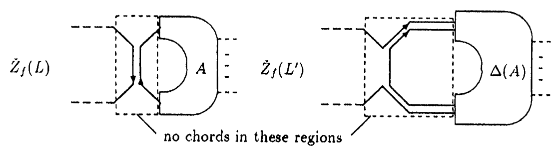

Fig. 3. Difference of $\check{Z}_f$ by the second Kirby move.

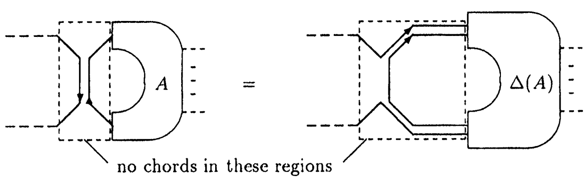

Fig. 4. Relation for second Kirby moves.

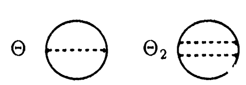

Fig. 5. Θ and Θ_2.

component.

Remark. $\Lambda'(L)$ is compatible with the Jones-Witten invariant in [6, 9].

2. Low degree part. We normalize a low degree part of $\Lambda'(L)$ for the KI moves.

Definition. Two elements D, D' in $\bar{\mathscr{A}}^{(l)}$ is called *stably equivalent* if $D \sqcup \Theta \sqcup \cdots \sqcup \Theta = D' \sqcup \Theta \sqcup \cdots \sqcup \Theta$ in $\bar{\mathscr{A}}^{(l+k)}$ for some $k \geq 0$, where Θ denote the chord diagram on a circle with one chord. Let $\bar{\mathscr{A}}_1^{(l)}$ denote the set of stable equivalence classes of $\bar{\mathscr{A}}^{(l)}$.

Proposition 2. *$\bar{\mathscr{A}}_1^{(l)}$ is a two dimensional space with basis $\{\Theta \sqcup \Theta \sqcup \cdots \sqcup \Theta, \Theta_2 \sqcup \Theta \sqcup \cdots \sqcup \Theta\}$, where Θ_2 denote the chord diagram on a circle with two chords as in Fig.2.*

For two elements $D_1 \in \bar{\mathscr{A}}_1^{(l_1)}$ and $D_2 \in \bar{\mathscr{A}}_1^{(l_1)}$, let $D_1 D_2$ be the image of $D_1 \sqcup D_2 \in \bar{\mathscr{A}}_1^{(l_1+l_2)}$. For $l_1, l_2 \geq 1$, $\bar{\mathscr{A}}_1^{(l_1)}$ and $\bar{\mathscr{A}}_1^{(l_2)}$ are isomorphic by identifying the corresponding basis. Let $\bar{\mathscr{A}}_1$ be a space spanned by e_0 and e_1 which correspond to $\Theta \sqcup \Theta \sqcup \cdots \sqcup \Theta$ and $\Theta_2 \sqcup \Theta \sqcup \cdots \sqcup \Theta$ respectively and we identity $\bar{\mathscr{A}}_1^{(l)}$ with $\bar{\mathscr{A}}_1$. Then $\bar{\mathscr{A}}_1$ has two dimensional algebra structure corresponding to the disjoint union, which is given by $e_0 e_0 = e_0$, $e_0 e_1 = e_1 e_0 = e_1$, and $e_1 e_1 = 0$. Note that $c_0 e_0 + c_1 e_1 \in \bar{\mathscr{A}}_1$ is invertible if and only if $c_0 \neq 0$. Let $\Lambda'_1(L)$ denote the image of $\Lambda'(L)$ in $\bar{\mathscr{A}}_1$. Then, for trivial knots $\infty_{\pm 1}$ with ± 1 framings, we have $\Lambda'_1(\infty_{+1}) = \frac{1}{2} e_0 + \frac{3}{8} e_1$, and $\Lambda'_1(\infty_{-1}) = -\frac{1}{2} e_0 + \frac{3}{8} e_1$. Hence $\Lambda'_1(\infty_{\pm 1})$ is invertible and we can modify $\Lambda'_1(L)$ for the KI moves as in the case of the Jones-Witten invariant. Let σ_+ (resp. σ_-) denote the number of positive (resp. negative) eigenvalues of the linking matrix of L, and $\Lambda_1(L) = \Lambda'_1(L)\Lambda'_1(\infty_{+1})^{-\sigma_+}\Lambda'_1(\infty_{-1})^{-\sigma_-}$. Let $\Lambda_{1,0}(L)$ and $\Lambda_{1,1}(L)$ be the coefficients of $\Lambda_1(L)$ as $\Lambda_1(L) = \Lambda_{1,0}(L) e_0 + \Lambda_{1,1}(L) e_1$.

For a framed link L and the corresponding three manifold M_L, we have the following.

Theorem 1. *$\Lambda_1(L)$ is an invariant of the three manifold M_L.*

Theorem 2. (1) $\Lambda_{1,0}(L) = |H_1(M_L)|$, *the order of the first homology group of M_L if $b_1(M_L) = 0$, and 0 if $b_1(M_L) > 0$, where $b_1(M_L)$ is the first Betti number of M_L.*

(2) $\Lambda_{1,1}(L) = -3\tilde{\lambda}(M_L)$, *where $\tilde{\lambda}(M_L)$ is Lescop's generalization* [5] *of the Casson-Walker invariant $\lambda(M_L)$ satisfying $\tilde{\lambda}(M_L) = |H_1(M_L)|\, \lambda(M_L)$ if $b_1(M_L) = 0$.*

Theorem 1 is a direct consequense of our construction of Λ_1. To prove Theorem 2, we use Ohtsuki's lemma given in [8, Corollary 2.5] and [7, Lemma 2.2]. According to Ohtsuki's lemma, we can restrict our attention to algebraically split links, for which we can prove (1) easily. To prove (2), adding to Ohtsuki's lemma, we use Lescop's formula which expresses the Casson-Walker invariant in terms of linking numbers and coefficients of the Conway polynomial [5]. For algebraically split links, this formula is rather simple and we can compare directly our invariant and their formula. Then we get (2).

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