Seminear-rings Characterized by their $\mathcal{S}$-ideals. II

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This paper is a continuation of the author's earlier paper [1]. For undefined terms and notations used here we refer to [1]. In section 1 we describe some properties of the lattice of $\mathcal{S}$-ideals of a distributively generated SI-seminear-ring (cf. [1]). In section 2 we define a topology in the space of all prime $\mathcal{S}$-ideals in a distributively generated SI-seminear-ring, and show that the subset consisting of all minimal prime $\mathcal{S}$-ideals forms a Hausdorff space. Below we announce our results, whose details will appear elsewhere. Only some indications of proof will be given to Theorems 3, 4.

1. Distributively generated SI-seminear-rings. Throughout this section R will denote a d.g. seminear-ring with an absorbing zero as defined in [1]. As remarked in [1], the product AB of $\mathcal{S}$-ideals A and B of R is an $\mathcal{S}$-ideal. Moreover, for each family of $\mathcal{S}$-ideals $\{A_i : i \in I\}$ of R, the sum $\sum_{i\in I} A_i$ as defined in [1], is the unique minimal member of the family of all $\mathcal{S}$-ideals of R containing the $\mathcal{S}$-ideals $\{A_i : i \in I\}$; and $\cap_{i\in I} A_i$ is the unique maximal member of the family of all $\mathcal{S}$-ideals of R contained in the $\mathcal{S}$-ideals $\{A_i : i \in I\}$. Using these facts, we may state Propositions 2.2 and 2.3 given in [1] in the following forms.

Proposition 1. *The following assertions are equivalent*:

(1) *R is SI.*

(2) *For each pair of $\mathcal{S}$-ideals A, B of R, $A \cap B = AB$.*

(3) *The set of $\mathcal{S}$-ideals of R (ordered by inclusion) is a semilattice $(\mathcal{L}_R, \Lambda)$ with $A \Lambda B = AB$ for each pair of $\mathcal{S}$-ideals A, B of R.*

Proposition 2. *The following assertions are equivalent*:

(1) *R is SI.*

(2) *The set of all $\mathcal{S}$-ideals of R (ordered by inclusion) forms a complete lattice $\mathcal{L}_R$ under the sum and intersection of $\mathcal{S}$-ideals with $I \cap J = IJ$ for each pair of $\mathcal{S}$-ideals I, J of R.*

We also have:

Proposition 3. *The following assertions are equivalent*:

(1) *For each pair of $\mathcal{S}$-ideals A, B of R, $A \cap B = AB$.*

(2) *R is SI.*

(3) *For each pair of $\mathcal{S}$-ideals A, B of R, $B \cap A = AB$.*

(4) *For each pair of $\mathcal{S}$-ideals A, B of R, $A \cap (A^{-1}B) = A \cap B$ $(A^{-1}B = \{r \in R : ra \in B$ for all $a \in A\})$.*

Next we show that the lattice $\mathcal{L}_R$ described in Proposition 2, is a (complete) Brouwerian and hence distributive lattice. A lattice $\mathcal{L}$ is called *Brouwerian* if for any $a, b \in \mathcal{L}$, the set of all $x \in L$ satisfying $a \wedge x \leq b$ contains a greatest element c, the *pseudo-complement* of a relative to b.

Proposition 4. *If R is an SI-seminear-ring, then the lattice $\mathcal{L}_R$ is distributive.*

Analogous to the notion of prime ideals in near-ring theory ([2], p. 62), we call an $\mathcal{S}$-ideal P of a seminear-ring R *prime* if $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$ holds for all $\mathcal{S}$-ideals I, J of R; P is called *completely prime* if for $a, b \in R$, $ab \in P \Rightarrow a \in P$ or $b \in P$; P *is minimal prime* if P is a minimal element of the set of prime $\mathcal{S}$-ideals of R. An $\mathcal{S}$-ideal K of R is *semiprime* if for all $\mathcal{S}$-ideals I of R, $I^2 \subseteq K \Rightarrow I \subseteq K$; K is *completely semiprime* if for $a \in R$ and n a positive integer, $a^n \in K \Rightarrow a \in K$. Furthermore, an $\mathcal{S}$-ideal Q of a seminear-ring R is called *irreducible* (*strongly irreducible*) if $I \cap J = Q \Rightarrow I = Q$ or $J = Q (I \cap J \subseteq Q \Rightarrow I \subseteq Q$ or $J \subseteq Q)$ holds for all $\mathcal{S}$-ideals I, J or R. Thus any prime $\mathcal{S}$-ideal is strongly irreducible and any strongly irreducible $\mathcal{S}$-ideal is irreducible. The following proposition shows that the concepts of prime, irreducible and strongly irreducible $\mathcal{S}$-ideals coincide for SI-seminear-rings.

Proposition 5. *Let R be an SI-seminear-ring. Then the following assertions for an $\mathcal{S}$-ideal P of R are equivalent*:

(1) *P is prime.*

(2) *P is irreducible.*

An $\mathcal{S}$-ideal J of R is a *direct summand* of R if there exists an $\mathcal{S}$-ideal J' called a *cosummand* of J, such that $J + J' = R$ and $J \cap J' = (0)$.

Proposition 6. *Let R be an SI-seminear-ring. Then the set of direct summands of R forms a Boolean sublattice of $\mathcal{L}_R$.*

The above proposition can be used to obtain the following characterization of distributively generated SI-seminear-rings.

Theorem 1. *The following assertions are equivalent:*

(1) *R is SI.*

(2) *Each proper $\mathcal{S}$-ideal of R is the intersection of prime $\mathcal{S}$-ideals which contain it.*

We now give an example of a class of regular seminear-rings, namely distributively generated regular seminear-rings which are neither (regular) near-rings nor (regular) semirings.

Example 1. Let R be a distributively generated regular zero symmetric (that is, having an absorbing zero) right near-ring (see [2], p. 407 for examples of such near-rings) and let D be the multiplicative subsemigroup of $(R, \cdot)$ which generates $(R, +)$. Furthermore, let $(S, \cdot)$ be a regular semigroup and let $\phi : (S, \cdot) \to (R, \cdot)$ be the homomorphism defined by $\phi(s) = 0$, for all $s \in S$. Let $A = S \cup R$. On the set A, introduce the structure of a right seminear-ring according to the procedure described in ([1], Example 1). Then $(A, +, \cdot)$ is a regular seminear-ring. Now adjoin an element $\theta \notin A$ to A, such that $a + \theta = \theta + a = a$ and $a\theta = \theta a = \theta$, for all $a \in A \cup \{\theta\}$. Let $A' = A \cup \{\theta\}$. Then $(A', +, \cdot)$ is a regular seminear-ring with an absorbing zero θ. Let $D' = S \cup D \cup \{\theta\}$. It is easily verified that D' is a multiplicative subsemigroup of $(A', \cdot)$, consisting of left distributive elements, which generates $(A', +)$. Hence A' is a d.g. regular (and hence SI) seminear-ring with an absorbing zero.

2. Prime $\mathcal{S}$-ideal spaces. Unless stated otherwise, R will denote a d.g. seminear-ring with an absorbing zero and a (multiplicative) identity, and P_R will denote the set of proper prime $\mathcal{S}$-ideals of R. Further for any $\mathcal{S}$-ideal I of R, we define the sets $\Theta_I = \{J \in P_R : I \not\subseteq J\}$ and $T(P_R) = \{\Theta_I : I \text{ is an } \mathcal{S}\text{-ideal of } R\}$.

Theorem 2. *Let R be an SI-seminear-ring. The set $T(P_R)$ constitutes a topology on the set P_R and the assignment $I \mapsto \Theta_I$ is a lattice isomorphism between the lattice $\mathcal{L}_R$ and the lattice of open subsets of P_R.*

The space P_R constructed in the above theorem need not be Hausdorff as shown by the following example.

Example 2. Let $R = \{0, a, 1\}$ with the following multiplication tables

$+$	0	a	1
0	0	a	1
a	a	a	a
1	1	a	1

$\cdot$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

Note that R is a reduced regular d.g. seminear-ring with an absorbing zero ; $\mathcal{L}_R = \{\{0\}, \{0, a\}, \{0, a, 1\}\}$ and $P_R = \{\{0\}, \{0, a\}\}$. The space of prime $\mathcal{S}$-ideals of R is clearly not Hausdorff.

Remark. If R is a regular near-ring with no nonzero nilpotent elements, then every R-subgroup of R is a (two-sided near-ring) ideal, all idempotents are central and every prime ideal is a minimal prime ideal (see [2], 9.158, 9.159, 9.163). The above example shows that unlike the situation in near-rings, prime ideals of a seminear-ring need not be minimal prime for regular seminear-rings with no nonzero nilpotent elements.

Next we shall prove that if R is a regular seminear-ring with central idempotents, then the subspace P_{OR} of P_R consisting of minimal prime $\mathcal{S}$-ideals is Hausdorff. For this purpose we need the following lemmas.

Lemma 1. *Let K be a completely semiprime $\mathcal{S}$-ideal of a (not necessarily d.g.) seminear-ring R. Then each of the following is true:*

(*i*) *If $ab \in K$ $(a, b \in R)$, then $ba \in K$.*

(*ii*) *If $ab \in K$ and $x \in R$, then $axb \in K$.*

(*iii*) *If $ab^n \in K$ (n is a positive integer), then $ab \in K$.*

(*iv*) *If $abc \in K$ $(a, b, c \in R)$ then $acb \in K$, and more generally, if $a_1a_2 \dots a_n \in K (a_i \in R$, $i = 1,2, \dots, n)$ then $a_{i_1}a_{i_2} \dots a_{i_n} \in K$ where $i_1, i_2, \dots, i_n$ is any permutation of $1,2, \dots, n$.*

Lemma 2. *Let R be a (not necessarily d.g.) reduced seminear-ring. If $a^n y = 0$ for some positive integer n and $a, y \in R$, then $ay = 0$.*

Definition 1. *A subset M of a seminear-ring R is called an m-system if for $a, b \in M$, there exists some $x \in R$ such that $axb \in M$.*

Lemma 3. *Let R be a reduced seminear-ring* *and let M be an m-system of R. If M does not intersect the completely semiprime $\mathscr{S}$-ideal K, then there exists an $\mathscr{S}$-ideal P which is maximal in the set of those completely semiprime $\mathscr{S}$-ideals which contain K and do not intersect M. Any such $\mathscr{S}$-ideal P is completely prime.*

Lemma 4. *Let R be a regular seminear-ring with central idempotents. If P is a prime $\mathscr{S}$-ideal of R, then $O_P = \{r \in R : ra = 0$ for some $a \notin P\}$ is an $\mathscr{S}$-ideal of R and $O_P \subseteq P$.*

The following theorem gives a useful characterization of minimal prime $\mathscr{S}$-ideals of regular seminear-rings with central idempotents.

Theorem 3. *Let R be a regular seminear-ring with central idempotents. A prime $\mathscr{S}$-ideal P is a minimal prime $\mathscr{S}$-ideal if and only if $P = O_P$.*

Sketch of proof. If $P \neq O_P$, there exists $a \in P \setminus O_P$ by Lemma 4 and $M = R \setminus P$ is an m-system. Put now
$K = \{a^{i_0}x_0a^{i_1}x_1 \cdots a^{i_n}x_na^{i_{n+1}} : n \in \mathbf{N} \cup \{0\}, i_0, i_{n+1} \in \mathbf{N} \cup \{0\}, i_1, \ldots, i_n \in \mathbf{N}, x_0, \ldots, x_n \in M\}$ (where $a^0 = 1$). Then $K \supseteq M$, $0 \notin K$ and K is an m-system. $O_P \cap K = \phi$ and O_P is completely semiprime. Lemma 3 implies that there exists a completely prime $\mathscr{S}$-ideal A such that $A \cap K = \phi$. As $A \subseteq P$ and $A \neq P$, P is not a minimal prime $\mathscr{S}$-ideal. The converse is clear. □

As an application of the above theorem, we can prove.

Theorem 4. *Let R be a regular seminear-ring with central idempotents. Then the subspace P_{OR} is Hausdorff.*

Sketch of proof. Let $P_1, P_2 \in P_{OR}$, $P_1 \neq P_2$. Then there exists $x \in P_1 \setminus P_2$. As $P_1 = O_{P_1}$ according to Theorem 3, we have $x \in O_{P_1}$, so there exists $t \notin P_1$ such that $xt = 0$. From Lemma 1(ii), we have $xRt = (0)$ and so $RxRt = (0)$. According to Proposition 2(i), Rx and Rt are $\mathscr{S}$-ideals of R, and from Proposition 1 $Rx \cap Rt = RxRt = (0)$. Thus $\Theta_{Rx} \cap \Theta_{Rt} = \Theta_{Rx \cap Rt} = \Theta_{(0)} = \phi$. As $P_1 \in \Theta_{Rt}$, $P_2 \in \Theta_{Rx}$, P_{OR} is Hausdorff. □

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References

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