

21. A Skorokhod Problem with Singular Drift and its Application to the Origin of Universes

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1. Introduction. Let $R(t)$ be strictly increasing and continuous in $t \geq 0$ with $R(0) = 0$. In a space-time domain

$$(1.1) \quad D = \{(t, x) ; t > 0, x \in [-R(t), R(t)]\},$$

we consider a singular diffusion equation and its formal adjoint

$$(1.2) \quad \frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} + \frac{x}{t} \frac{\partial u}{\partial x} = 0, \quad -\frac{\partial \mu}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 \mu}{\partial x^2} - \frac{\partial}{\partial x} \left(\frac{x}{t} \mu \right) = 0,$$

with the reflecting boundary condition. (1.2) determines a transition probability $Q(s, x ; t, dy)$, $s, t \in [a, b]$, $0 < a < b < \infty$. Since $\{Q(s, x ; t, dy) ; s \in (0, \varepsilon]\}$ is tight because of (1.1), we can choose $\xi(s) \downarrow 0$ so that

$$(1.3) \quad Q^\xi(0, 0 ; t, dy) = \lim_{s \downarrow 0} Q(s, \xi(s) ; t, dy)$$

exists, but the limit $Q^\xi(0, 0 ; t, dy)$ depends on ξ and is not uniquely determined in general. We will discuss this problem and its implication to the origin of universes in terms of a Skorokhod problem with singular drift x/t .

2. A Skorokhod problem. Instead of (1.2) with the moving reflecting boundary we consider a two-sided Skorokhod problem with singular drift

$$(2.1) \quad X_t = \sigma \beta_t + \int_0^t \frac{X_s}{s} ds + \Phi_t, \quad |X_t| \leq R(t),$$

where β_t denotes a one-dimensional Brownian motion, and

$$(2.2) \quad \Phi_t \text{ is continuous in } t \geq 0, \Phi_0 = 0,$$

$$\Phi_t = \Phi_t^{(-)} - \Phi_t^{(+)}, \text{ for } t > 0,$$

$$\Phi_t^{(-)} \text{ increases only on } \{s : X(s) = -R(s)\},$$

$$\Phi_t^{(+)} \text{ increases only on } \{s : X(s) = R(s)\}.$$

We will construct solutions of the problem (2.1), and show that the shape of the boundary of the domain D influences the uniqueness and non-uniqueness of solutions of (2.1). Assuming

$$(2.3) \quad R(t) = (\alpha t)^\gamma, \quad 0 < \gamma < 1, \text{ for small } t,$$

where $\alpha > 0$, we shall analyze the behaviour of solutions near the origin.

3. The case without boundary. Equation (1.2) but $[a, b] \times \mathbf{R}$ without boundary determines another transition probability $P(s, x ; t, dy)$. Contrary to the case with reflecting boundary, $P(0, 0 ; t, dy)$ cannot be well-defined, since $\{P(s, x ; t, dy) : s \in (0, \varepsilon]\}$ is not tight. Hence, a stochastic differential equation

$$(3.1) \quad X_t = \sigma \beta_t + \int_0^t \frac{X_s}{s} ds$$

has no adapted solution, where β_t denotes a one-dimensional Brownian motion. Nevertheless, a theorem of Jeulin-Yor [5] (cf. [6]) claims that X_t satisfies

(3.1) if and only if

$$(3.2) \quad X_t = \sigma B_t + tY,$$

where B_t is given by

$$(3.3) \quad B_t = -t \int_t^\infty \frac{d\beta_s}{s},$$

which is a one-dimensional Brownian motion, and Y is a random variable. As a matter of fact, (3.1) is an equation derived from the second time reversal.

4. The minimum and maximum solutions. Avoiding the singularity at the origin, we consider a problem starting from the lower boundary $-R(\varepsilon)$:

$$(4.1) \quad X_t = -R(\varepsilon) + \sigma(\beta_t - \beta_\varepsilon) + \int_0^t \frac{X_s}{s} ds + \Phi_t, \quad |X_t| \leq R(t), \quad t \geq \varepsilon,$$

subject to (2.2) with $\Phi_\varepsilon = 0$, where $\varepsilon > 0$. Since there is no singularity in the problem (4.1), it has a unique solution.

Lemma 4.1 (i) Let $X_t^{(-\varepsilon)}$ be the solution of (4.1). If $0 < \varepsilon' < \varepsilon$, then

$$(4.2) \quad X_t^{(-\varepsilon)} \leq X_t^{(-\varepsilon')}, \quad \text{for } t \geq \varepsilon,$$

namely $X_t^{(-\varepsilon)}$ is monotone increasing as $\varepsilon \downarrow 0$. (ii) There exists

$$(4.3) \quad \underline{X}_t = \lim_{\varepsilon \downarrow 0} X_t^{(-\varepsilon)},$$

which is the minimum solution of the problem (2.1).

Proof. Let us define $T = \inf \{t : X_t^{(-\varepsilon)} = X_t^{(-\varepsilon')}\}$. Then, (4.2) holds, since

$$(4.4) \quad X_t^{(-\varepsilon)} < X_t^{(-\varepsilon')}, \quad \text{for } \varepsilon \leq t < T, \quad \text{and } X_t^{(-\varepsilon)} = X_t^{(-\varepsilon')}, \quad \text{for } t \geq T.$$

Taking limit $\varepsilon \downarrow 0$ in (4.1), we have the second assertion.

We consider also a problem starting from the upper boundary $R(\varepsilon)$ at $\varepsilon > 0$:

$$(4.5) \quad X_t = R(\varepsilon) + \sigma(\beta_t - \beta_\varepsilon) + \int_\varepsilon^t \frac{X_s}{s} ds + \Phi_t, \quad |X_t| \leq R(t), \quad t \geq \varepsilon,$$

subject to (2.2) with $\Phi_\varepsilon = 0$. Then, we have

Lemma 4.2. (i) Let $X_t^{(\varepsilon)}$ be the solution of (4.5). If $0 < \varepsilon' < \varepsilon$, then

$$(4.6) \quad X_t^{(\varepsilon)} \geq X_t^{(\varepsilon')}, \quad \text{for } t \geq \varepsilon,$$

namely $X_t^{(\varepsilon)}$ is monotone decreasing as $\varepsilon \downarrow 0$. (ii) There exists

$$(4.7) \quad \overline{X}_t = \lim_{\varepsilon \downarrow 0} X_t^{(\varepsilon)},$$

which is the maximum solution of the problem (2.1).

Theorem 4.1. Let \underline{X}_t (resp. \overline{X}_t) be defined by (4.4) (resp. (4.7)), and X_t be any solution of the problem (2.1). Then

$$(4.8) \quad \underline{X}_t \leq X_t \leq \overline{X}_t, \quad \text{for } t \geq 0,$$

for almost all Brownian paths. The minimum solution \underline{X}_t (resp. maximum one \overline{X}_t) reaches the lower-(resp. upper-)boundary immediately.

5. The uniqueness and non-uniqueness of solutions. First of all we remark that for a one-dimensional Brownian motion B_t , the law of iterated logarithm holds; namely, for almost all Brownian paths

$$(5.1) \quad \lim_{t \downarrow 0} \frac{\overline{B}_t}{\sqrt{2t \log \log t^{-1}}} = 1.$$

Theorem 5.1. *Let $R(t)$ be given in (2.3). Then, there exist solutions of the Skorokhod problem (2.1). If $0 < \gamma < 1/2$, then solutions of (2.1) are not uniquely determined, while the uniqueness holds, if $1/2 \leq \gamma < 1$.*

Proof. Because of (5.1), if $1/2 \leq \gamma < 1$, any solution of (2.1) hits the lower and upper boundaries $\{-R(t), R(t)\}$ immediately, as do the minimum and maximum solutions \underline{X}_t and \overline{X}_t . Therefore, the uniqueness of solutions holds. Moreover, (5.1) implies that if $0 < \gamma < 1/2$, then a Brownian motion B_t does not immediately hit the boundary $\{-R(t), R(t)\}$, and hence processes $X_t = \sigma B_t + tY$ also do not immediately hit the boundary. Therefore, if we define a process by

$$(5.2) \quad X_t = \sigma B_t + tY, \text{ for } t < \varepsilon,$$

with B_t given in (3.3), where $\varepsilon > 0$ is the first hitting time to the boundary, and for $t \geq \varepsilon$ by a solution of a Skorokhod problem

$$(5.3) \quad X_t = \sigma B_\varepsilon + \varepsilon Y + \sigma(\beta_t - \beta_\varepsilon) + \int_\varepsilon^t \frac{X_s}{s} ds + \Phi_t, \quad |X_t| \leq R(t),$$

then the process solves the problem (2.1).

6. Another representation of solutions of (2.1). Corresponding to Jeulin-Yor's theorem, we have, in the case of the moving reflecting boundary,

Theorem 6.1. *Assume*

$$(6.1) \quad \lim_{t \rightarrow \infty} \frac{R(t)}{t} = 0.$$

Then, X_t satisfies equation (2.1), if and only if

$$(6.2) \quad X_t = \sigma \beta_t - t \int_t^\infty \frac{d\Phi_s}{s},$$

where B_t is a one-dimensional Brownian motion given in (3.3).

Proof. Let X_t satisfy equation (2.1) and B_t be defined by (3.3). Then, we have, with integration by parts formula,

$$(6.3) \quad \int_t^\infty \frac{X_s}{s^2} ds = \frac{X_t}{t} + \int_t^\infty \frac{dX_s}{s},$$

since $|X_r/r| \leq R(r)/r \rightarrow 0$, as $r \rightarrow \infty$ by (6.1). Therefore,

$$(6.4) \quad \begin{aligned} \sigma B_t &= -t \int_t^\infty \frac{d\sigma \beta_s}{s} = -t \int_t^\infty \frac{dX_s}{s} + t \int_t^\infty \frac{X_s}{s^2} ds + t \int_t^\infty \frac{d\Phi_s}{s}, \\ &= X_t + t \int_t^\infty \frac{d\Phi_s}{s}. \end{aligned}$$

Thus X_t satisfies (6.2). The converse can be verified in the same way.

The representation (6.2) means that if one looks at the process X_t knowing the future (the second time reversal), the singular drift field in (2.1) disappears.

7. A central solution. When $0 < \gamma < 1/2$, taking a solution with $Y \equiv 0$ in (5.2) and (5.3), we call it a "central solution" or "central process". It is central, in the sense that it starts from the origin as a Brownian motion. This means that in the representation (6.2) the integral term vanishes for sufficiently small t , namely, there is $\varepsilon > 0$ such that

$$(7.1) \quad t \int_t^\infty \frac{d\Phi_s}{s} = 0, \text{ for } t \leq \varepsilon.$$

Theorem 7.1. Assume $R(t) = (\alpha t)^\gamma$, $0 < \gamma < 1/2$, for small t , and let X_t^0 be the central solution. If $0 < \gamma \leq 1/2$ for large t , then $X_t - X_t^0 \rightarrow 0$ in law as $t \uparrow \infty$ for any solution X_t of the problem (2.1); if $1/2 < \gamma < 1$ for large t , then $\overline{X}_t - X_t^0$ (resp. $\underline{X}_t - X_t^0$) does not converge in law as $t \uparrow \infty$, where \overline{X}_t (resp. \underline{X}_t) is the maximum (resp. minimum) solution of (2.1).

A proof can be carried over with the help of the law of iterated logarithm:

$$(7.2) \quad \overline{\lim}_{t \uparrow \infty} \frac{B_t}{\sqrt{2t \log \log t}} = 1.$$

8. An application. Applicability of diffusion theory to various fields of different orders of magnitude, including quantum physics, in particular particle theory, biology, cosmology, and so on, was discussed in (cf. [3], Section 4.7 and Chapter 9). Let D be the space-time domain given in (1.1). We consider a simple model of a one-dimensional universe after ([2], personal communication). We consider a diffusion equation and its formal adjoint in D

$$(8.1) \quad \frac{\partial u}{\partial t} + \frac{1}{2} \bar{\sigma}^2 \frac{\partial^2 u}{\partial x^2} + a(t, x) \frac{\partial u}{\partial x} = 0, \quad -\frac{\partial \mu}{\partial t} + \frac{1}{2} \bar{\sigma}^2 \frac{\partial^2 \mu}{\partial x^2} - \frac{\partial}{\partial x} (a(t, x) \mu) = 0,$$

with the moving reflecting boundary condition, where a drift field $a(t, x)$ is given, taking into account of the Hubble law, through

$$(8.2) \quad a(t, x) = \frac{R'(t)}{R(t)} x, \text{ where } 0 \leq \frac{x}{R(t)} \leq 1, \text{ for } t > 0.$$

If we assume (2.3), then

$$(8.3) \quad a(t, x) = \gamma \frac{x}{t}.$$

Theorems 4.1, 5.1, 6.1, and 7.1 solve this problem. In fact, through time change, with a new diffusion coefficient $\sigma = \bar{\sigma}/\sqrt{\gamma}$, equation (8.1) with (8.3) reduces to equation (1.2). Therefore, it is enough to consider solutions of the two-sided Skorokhod problem (2.1) instead of (8.1).

The reason why we adopt a diffusion process as a model of the universe should be explained. It is based on a theorem (cf. [3], Chapter 4), which claims the equivalence between the diffusion equations (8.1) and the Schrödinger equation

$$(8.4) \quad i \frac{\partial \phi}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 \phi}{\partial x^2} - V(t, x) \phi = 0,$$

where $\phi = \exp(R + iS)$. As a matter of fact, for the equivalence we need the so-called Schrödinger representation (which can be regarded as a sort of "the equation of motion") of diffusion processes:

$$(8.5) \quad \frac{\partial \phi}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 \phi}{\partial x^2} + c(t, x) \phi = 0, \quad -\frac{\partial \hat{\phi}}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 \hat{\phi}}{\partial x^2} + c(t, x) \hat{\phi} = 0,$$

where $\phi = \exp(R + S)$ and $\hat{\phi} = \exp(R - S)$. Then, we have

$$(8.6) \quad c + V + (\sigma S_x)^2 + 2\dot{S} = 0.$$

Because of the Hubble law (8.2) we choose

$$(8.7) \quad \phi(t, x) = \eta(t) \exp\left(\frac{1}{2\sigma^2} \frac{R'(t)}{R(t)} x^2\right) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(\frac{x^2}{2\sigma^2 t}\right).$$

Then, "the creation and killing $c(t, x)$ induced by $\phi(t, x)$ " vanishes, i.e.,

$$-c\phi = \frac{\partial\phi}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2\phi}{\partial x^2} = 0,$$

in other words, $\phi(t, x)$ is a space-time harmonic function. Therefore, in our case formula (8.6) implies

$$(8.8) \quad V = -(\sigma S_x)^2 - 2\dot{S}.$$

Moreover, because of the duality relation, we have

$$S(t, x) = \log \phi(t, x) - \frac{1}{2} \log \mu(t, x).$$

Since, for small t and x , and $0 < \gamma < 1/2$,

$$\mu(t, x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{x^2}{2\sigma^2 t}\right),$$

together with (8.7) we have,

$$(8.9) \quad V(t, x) = -\frac{3}{4} \frac{1}{\sigma^2} \frac{x^2}{t^2} + \frac{1}{2} \frac{1}{t}, \text{ for small } t > 0,$$

for the potential field in (8.4).

In other words, with a diffusion process on D , three equations of different types (1.2), (8.4), and (8.5) are associated, they are equivalent each other, and for the distribution density $\mu(t, x)$ of the diffusion process we have

$$(8.10) \quad \mu(t, x) = \phi(t, x) \hat{\phi}(t, x) = \phi(t, x) \bar{\phi}(t, x),$$

which is one of the fundamental formulae of time reversal in diffusion theory, and also in quantum mechanics (cf. [3], Chapter 4 for details).

Another important point should be emphasized here. If we are to describe the behaviour of radiation and mass "particles" in the universe, we are dealing with a many-particle system, not with a single sample path of an ordinary diffusion process. The distribution density $\mu(t, x)$ of our diffusion process should therefore be viewed as the spatial statistical distribution density of (infinitely) many interacting particles; more precisely, we can regard it as the limiting distribution in the sense of the propagation of chaos, as n tends to infinity, of a system of n -interacting particles (cf. [3], Chapter 8). This point, together with interference of superposition, is important in connection with the problem of seeds for the large scale structure of the universe in terms of the fluctuation of the distribution (cf. [1]).

In short, with the terminology of conventional physics, for our model we adopt quantum mechanics (notice that "imaginary time" is not employed) in describing the universe, in particular near its origin (0,0), and moreover we assume that $R(t)$ is determined by the general theory of relativity, possibly through the Friedman equation

$$(8.11) \quad \left(\frac{R'}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi}{3} G\rho + \frac{\Lambda}{3},$$

where $\rho \approx R^{-4}$ if the universe is in the radiation dominated phase, while

$\rho \approx R^{-3}$ if matter dominated.

9. Interpretation. Despite the singularity of the drift field, universes can naturally start at the origin $(0,0)$ from nothing. If we adopt a boundary-less model of universes in one-dimension. Namely $(1+1)$ -dimensional model, the spatial distribution is on a circle of radius $R(t)$, which is viewed as the closed interval $[-R(t), R(t)]$, for $t > 0$, in our model. Namely, we identify $\pi - \theta$ with θ for $\theta \in [-\pi/2, \pi/2]$ in the cylindrical coordinates in three-dimensions. Then, introducing a new variable $x = 2R(t)\theta/\pi \in [-R(t), R(t)]$, we get our space time domain $D = \{(t, x) ; t > 0, x \in [-R(t), R(t)]\}$. Accordingly, \overline{X}_t (resp. \underline{X}_t) can be interpreted as a universe with spin $+$ (resp. spin $-$). From $t = 0$ to a critical time $t_0 > 0$ (which might possibly be the Planck time $= 5.39 \times 10^{-44}$ sec) there is still a considerable amount of concentration of radiation near the origin if $t < t_0$, but it will quickly spread out and be homogenized. We have assumed $R(t) = (\alpha t)^\gamma$, $0 < \gamma < 1$, for small t , but $0 < \gamma < 1/2$ is suggested in order to retain enough fluctuation as seeds for large scale structure. If $0 < \gamma < 1/2$ for small t , our universe is possibly a superposition of solutions of Schrödinger equation (8.4) (or (8.1)), for this cf. [4].

If we take into account of the so-called inflation model of universes, $R(t)$ might possibly be

$$(9.1) \quad R(t) = (\alpha(t \wedge t_1))^\gamma \exp \{ \kappa((t - t_1)^+ \wedge \tau) \} + (\alpha_2(t - t_2)^+)^{\gamma_2},$$

in the radiation dominated phase, where $0 < t_0 \leq t_1 < t_2$, $\tau = t_2 - t_1$ is the time span of inflation, t_2 is the moment that the so-called big bang occurs, and $\gamma_2 = 1/2$. During the inflation, we have a drift field

$$(9.2) \quad a(t, x) = \kappa x, \text{ for } t \in [t_1, t_2],$$

through (8.2). If the universe becomes matter dominated, then $R(t) \approx t^{2/3}$.

Our $(1+1)$ -dimensional model can be generalized to a $(1+3)$ -dimensional model.

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