20. *Some Estimates for Eigenvalues of Schrödinger Operators*

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1. Introduction. In this paper we give estimates for large eigenvalues of Schrödinger operators $-\Delta + V$ with increasing potential V. Let $N(\lambda)$ be the number of eigenvalues of the Schrödinger operator less than λ. Under some conditions on V we can prove the asymptotic formula

(1) $N(\lambda) \sim (2\pi)^{-d} \mid \{(\xi, x) \in \mathbf{R}^d \times \mathbf{R}^d : |\xi|^2 + V(x) < \lambda\} \mid (\lambda \to \infty)$,

which means that there is a correspondence between each eigenvalue less than λ and each set with volume $(2\pi)^d$ in $\{(\xi, x) \in \mathbf{R}^d \times \mathbf{R}^d : |\xi|^2 + V(x) < \lambda\}$. This correspondence is known as the Bohr-Sommerfeld quantization rule. A lot of people study the conditions on potentials for the formula (1), for instance, Feigin [3], Fleckinger [4], Rozenbljum [5], Simon [6], Tachizawa [7], Titchmarsh [8] and so on.

In this paper we give another formulation of this problem. Let $A = (\mathbf{N} \times \mathbf{Z}) \cup \{(0,2n') : n' \in \mathbf{Z}\}$ and $B = \{(m, n) : m = (m_1, \ldots, m_d), n = (n_1, \ldots, n_d), (m_i, n_i) \in A, i = 1, \ldots, d\}$. Our claim is that there is a correspondence between each eigenvalue and each point $(2\pi m, n)$ for $(m, n) \in B$. Let $\theta_{m,n} = |2\pi m|^2 + V(n/2)$ for $(m, n) \in B$ and $\{\mu_k\}_{k \in \mathbf{N}}$ the rearrangement of $\{\theta_{m,n}\}_{(m,n) \in B}$ in the nondecreasing order. We show that

(2) $$\lim_{k \to \infty} \frac{\lambda_k}{\mu_k} = 1$$

under some conditions on V. The formula (2) gives a relation between the asympototic behavior of eigenvalues and the symbol of the Schrödinger operator, which is a new result.

The class of the potentials V studied in this paper contains slowly increasing ones, for example, $V(x) = \log \cdots \log |x|$ (large $|x|$). The formula (1) is proved in [7] for radial, slowly increasing potentials. But it is not known whether the formula (1) holds or not for non-radial slowly increasing potentials. Our theorem gives a new approach to the study of eigenvalues of Schrödinger operators with slowly increasing potentials.

2. Theorem. We consider potentials $V(x)$ satisfying the following conditions.

(H1) $V \in C^\infty(\mathbf{R}^d)$, $V \geq 1$, $V(x) \to \infty$ $(|x| \to \infty)$.

(H2) There are positive constants c, γ such that
$$V(x + y) \leq c(1 + |y|)^\gamma V(x) \quad (x, y \in \mathbf{R}^d).$$

(H3) There is a constant τ, $1/2 \leq \tau < 1$, such that, for every $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{Z}^d_+$, $1 \leq |\alpha| = \alpha_1 + \cdots + \alpha_d$,
$$|\partial_x^\alpha V(x)| \leq C_\alpha V(x)^\tau \quad (x \in \mathbf{R}^d)$$
where C_α is a positive constant depending only on α.

We consider the Schrödinger operator $-\Delta + V$ on $C_0^\infty(\mathbf{R}^d)$ where $\Delta = \sum_{i=1}^d \partial^2/\partial x_i^2$. Let L be the selfadjoint realization of $-\Delta + V$ in $L^2(\mathbf{R}^d)$ and $D(L)$ the domain of L. By the condition (H1), L has only discrete spectrum $\lambda_1 < \lambda_2 \leq \cdots$ (cf. [2], [8]). We define $\tilde{\theta}_{m,n} = \{|2\pi m|^2 + V(n/2)\}^\tau$ for $(m, n) \in B$ and $\tilde{\mu}_k = \tilde{\theta}_{m,n}$ for $\mu_k = \theta_{m,n}$.

We have the following theorem.

Theorem 2.1. *Suppose that a real valued function V satisfies the conditions* (H1), (H2) *and* (H3). *Then there are positive constants C, K such that*

$$\mu_k - C\tilde{\mu}_k \leq \lambda_k \leq \mu_k + C\tilde{\mu}_k$$

for all $k \geq K$.

As a corollary of Theorem 2.1, we have the following result.

Corollary 2.1. *Under the same assumptions as in the previous theorem, we have*

$$\lim_{k\to\infty} \frac{\lambda_k}{\mu_k} = 1.$$

3. Outline of the proof of Theorem 1.1. For the proof of Theorem 2.1, we use the Wilson basis. In [1] Daubechies, Jaffard and Journé discovered a function $\phi(t)$ which satisfies the following conditions.

(a) $\phi(t)$ is real, even function in $\mathscr{S}(\mathbf{R})$ and $\hat{\phi}(s) = (2\pi)^{-1/2} \int_{\mathbf{R}} \phi(t) e^{-ist}\, dt = 2\sqrt{\pi}\phi(4\pi s)$.

(b) There exist η, $C > 0$ such that $|\phi(t)| \leq Ce^{-\eta|t|}$ for all $t \in \mathbf{R}$.

(c) Let $\hat{\psi}_{m,n}(\xi) = c_m\{\phi(\xi - 2\pi m) + (-1)^{m+n}\phi(\xi + 2\pi m)\}e^{-in\xi/2}$ for $m \in \mathbf{Z}_+$, $n \in \mathbf{Z}$, $\xi \in \mathbf{R}$, where $c_m = 1/\sqrt{2}$ for $m \geq 1$ and $c_0 = 1/2$. Then $\{\psi_{m,n}(x)\}_{(m,n)\in A}$ is an orthonormal basis in $L^2(\mathbf{R})$.

For $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$, $m = (m_1, \ldots, m_d) \in \mathbf{Z}_+^d$, $n = (n_1, \ldots, n_d) \in \mathbf{Z}^d$, we define

$$\psi_{m,n}(x) = \psi_{m_1,n_1}(x_1) \times \cdots \times \psi_{m_d,n_d}(x_d).$$

We can easily prove that $\{\psi_{m,n}\}_{(m,n)\in B}$ is an orthonormal basis in $L^2(\mathbf{R}^d)$. We call $\{\psi_{m,n}\}_{(m,n)\in B}$ the Wilson basis in $L^2(\mathbf{R}^d)$.

We have the following Lemmas.

Lemma 3.1. *Suppose that a real valued function V satisfies the conditions* (H1), (H2) *and* (H3). *Then there is a positive constant C such that*

$$\sum_{(m,n)\in B} |(f, \psi_{m,n})|^2 \{\theta_{m,n} - C\tilde{\theta}_{m,n}\} \leq (Lf, f)$$
$$\leq \sum_{(m,n)\in B} |(f, \psi_{m,n})|^2 \{\theta_{m,n} + C\tilde{\theta}_{m,n}\}$$

for all $f \in D(L)$ where $(\cdot,\cdot)$ denotes the inner product of $L^2(\mathbf{R}^d)$.

Lemma 3.2. *Suppose that a real valued function V satisfies the condition* (H1). *The k-th eigenvalue of the selfadjoint operator L is characterized by the following formulas:*

$$\lambda_k = \sup_{M_{k-1}} \inf\{(Lu, u) : u \in D(L), \|u\| = 1, u \perp M_{k-1}\},$$
$$\lambda_k = \inf_{M_k \subset D(L)} \sup\{(Lu, u) : \|u\| = 1, u \in M_k\}$$

where M_k denotes a k-dimensional subspace of $L^2(\mathbf{R}^d)$ and inf *and* sup *are taken over all $k-1$ and k-dimensional subspaces, respectively.*

The proof of Lemma 3.2 is given in [9].

For $\mu_k = \theta_{m,n}$ we set $\phi_k = \psi_{m,n}$. Let M_k be the subspace of $L^2(\mathbf{R}^d)$ which is spanned by $\{\phi_1, \ldots, \phi_k\}$.

By Lemma 3.1, we have

$$\mu_k - C\tilde{\mu}_k \leq \inf\{(Lu, u) : u \in D(L), \|u\| = 1, u \perp M_{k-1}\},$$

for sufficiently large k.

Hence the first characterization in Lemma 3.2 gives

$$\mu_k - C\tilde{\mu}_k \leq \lambda_k$$

for all sufficiently large k.

Similarly we have

$$\mu_k + C\tilde{\mu}_k \geq \sup\{(Lu, u) : \|u\| = 1, u \in M_k\}$$

and we get

$$\lambda_k \leq \mu_k + C\tilde{\mu}_k.$$

Therefore Theorem 2.1 is proved.

The proof of Lemma 3.1 is given by the following lemma and an elementary calculus.

Lemma 3.3. *Suppose that a real valued function* V *satisfies the conditions* (H1), (H2) *and* (H3). *For every* $\alpha, \beta \in \mathbf{Z}_+$, *there exists a constant* $C = C(\alpha, \beta) > 0$ *such that*

$$|(L\psi_{m,n}, \psi_{m',n'}) - \theta_{m,n}(\psi_{m,n}, \psi_{m',n'})| \leq C\frac{(\tilde{\theta}_{m,n}, \tilde{\theta}_{m',n'})^{1/2}}{(1+|m-m'|^2)^{\alpha}(1+|n-n'|^2)^{\beta}}$$

for all $m, m' \in \mathbf{Z}_+^d$, $n, n' \in \mathbf{Z}^d$.

In the proof of Lemma 3.3 we use the assumptions on V and the exponential decay property of the Wilson basis.

The detail will appear elsewhere.

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