68. *Demjanenko Matrix for Imaginary Abelian Fields of Odd Conductors*

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1. Introduction. In [2] and [3], Sands and Schwarz investigated an interesting relation between the determinant of the Demjanenko matrix and the relative class numbers of imaginary abelian fields of prime power conductors. It is the main purpose of this paper to generalize their results to the case of imaginary abelian fields *of odd conductors*.

Fix a positive odd integer m. Let $G = (\boldsymbol{Z}/m\boldsymbol{Z})^*$ and $S = \left\{\bar{a} = a + \boldsymbol{Z} \in G \mid 1 \leq a \leq \frac{m}{2}\right\}$. We consider an imaginary subfield K of the m-th cyclotomic field $\boldsymbol{Q}(\zeta_m)$. Then, by Galois theory, K corresponds to a subgroup N of G with $-\bar{1} \notin N$. We denote by R a system of representatives for the Galois group $G/(N\cdot\{\pm\bar{1}\})$ of the maximal real subfield K^+ of K. For any element $\bar{a}$ of G, define the integers $C(\bar{a})$ and $C'(\bar{a})$ by

$$C(\bar{a}) = |\bar{a}N \cap S|,$$

$$C'(\bar{a}) = |\bar{a}N \cap -S| = |N| - C(\bar{a}).$$

Let us define the generalized Demjanenko matrix $D_{m,N}$ by

$$D_{m,N} = (C(\bar{a}\bar{b}) - C'(\bar{b}))_{\bar{a},\bar{b}\in R}.$$

Let E_K (*resp.* E_K^+) be the group of units in K (*resp.* K^+) and W_K the group of all roots of unity in K. We call $Q_K = [E_K : W_K E_K^+]$ the *unit index* of K. Let X^- be the set of odd Dirichlet characters mod m which are trivial on N. We prove the following theorem:

Theorem. *We have*

$$\det D_{m,N} = \pm \frac{2}{Q_K |W_K|} \cdot \prod_{\chi\in X^-} \prod_{p|m} (1 - \chi(\bar{p})) \cdot \prod_{\chi\in X^-} (2 - \chi(\bar{2})) \cdot h^-(K),$$

where p's are primes dividing m and $h^-(K)$ is the relative class number of K.

Remark. Let g be the number of primes in K lying above an odd prime p and f be the residue class degree of primes in K lying above 2. Furthermore, let $\mathfrak{p}_p$ and $\mathfrak{p}_2$ be primes in K^+ lying above p and 2 respectively. Then we have

$$\prod_{\chi\in X^-} (1 - \chi(\bar{p})) = \begin{cases} 1 & \cdots \ \mathfrak{p}_p \text{ ramifies in } K \\ 2^g & \cdots \ \mathfrak{p}_p \text{ remains prime in } K \\ 0 & \cdots \ \mathfrak{p}_p \text{ splits in } K \end{cases}$$

and

$$\prod_{\chi\in X^-} (2 - \chi(\bar{2})) = \begin{cases} (2^{f/2} + 1)^{2n/f} & \cdots \ \mathfrak{p}_2 \text{ remains prime in } K \\ (2^f - 1)^{n/f} & \cdots \ \mathfrak{p}_2 \text{ splits in } K \end{cases}$$

(For details, see [5] p.24.)

Corollary (Sands and Schwarz). If m is an odd prime power, we have

$$\det D_{m,N} = \pm \frac{2}{|W_K|} \cdot \prod_{\chi\in X^-} (2-\chi(\bar{2}))\cdot h^-(K),$$

where $h^-(K)$ is the relative class number of K.

2. Proof of the Theorem. we define the integer $r(\bar{a})$ $(\bar{a} \in G)$ by

$$r(\bar{a}) \equiv a \bmod m,\ 0 \leq r(\bar{a}) < m.$$

The analytic class number formula for $h^-(K)$ is

$$h^-(K) = Q_K\,|W_K| \prod_{\chi\in X^-} \left(-\frac{1}{2}B_{1,\chi}\right),$$

where $B_{1,\chi}$ is the generalized Bernouilli number ([5]). We recall the following equation

$$\frac{1}{m}\sum_{\bar{a}\in G} r(\bar{a})\chi(\bar{a}) = \prod_{p\mid m}(1-\chi(\bar{p}))\cdot B_{1,\chi}$$

(See [4]). From these two equations, we obtain

$$\frac{1}{2^n}\prod_{\chi\in X^-}\sum_{\bar{a}\in G} r(\bar{a})\chi(\bar{a}) = \frac{(-m)^n}{Q_K\,|W_K|}\cdot\prod_{\chi\in X^-}\prod_{p\mid m}(1-\chi(\bar{p}))\cdot h^-(K),$$

where $n = |R|$. On the other hand, it is easily proved that

$$\prod_{\chi\in X^-}\sum_{\bar{a}\in G} r(\bar{a})\chi(\bar{a})\cdot\prod_{\chi\in X^-}(2-\chi(\bar{2})) = (-m)^n\prod_{\chi\in X^-}\sum_{\bar{a}\in S}\chi(\bar{a}).$$

So, we get

$$\prod_{\chi\in X^-}\sum_{\bar{a}\in S}\chi(\bar{a}) = \frac{2^n}{Q_K\,|W_K|}\cdot\prod_{\chi\in X^-}\prod_{p\mid m}(1-\chi(\bar{p}))\cdot\prod_{\chi\in X^-}(2-\chi(\bar{2}))\cdot h^-(K).$$

Fix a choice of $\omega \in X^-$ and define $\delta : G \to \{\pm 1\}$ by

$$\delta(a) = \begin{cases} 1 & \text{if } a \in S \\ -1 & \text{if } a \in -S \end{cases}.$$

We denote by X^+ the set of even Diriclet characters mod m which are trivial on N. Then, we have

$$\begin{aligned}\prod_{\chi\in X^-}\sum_{\bar{a}\in S}\chi(\bar{a}) &= \prod_{\chi\in X^-}\sum_{\bar{a}\in S}\chi(\bar{a})\delta(\bar{a}) \\ &= \prod_{\psi\in X^+}\sum_{\bar{a}\in S}\psi(\bar{a})\omega\delta(\bar{a}).\end{aligned}$$

Note that ψ and $\omega\delta$ are well-defined on $G/(N\cdot\{\pm\bar{1}\})$ and $G/\{\pm\bar{1}\}$ respectively. We put

$$f(A) = \sum_{\pm\bar{a}\in A}\omega\delta(\pm\bar{a})$$

for $A \in G/(N\cdot\{\pm\bar{1}\})$. By using the Dedekind determinant formula ([1] p. 89),

$$\begin{aligned}\prod_{\chi\in X^-}\sum_{\bar{a}\in S}\chi(\bar{a}) &= \prod_{\psi\in X^+}\sum_{\pm\bar{a}\in G/\{\pm\bar{1}\}}\psi(\pm\bar{a})\omega\delta(\pm\bar{a}) \\ &= \prod_{\psi\in X^+}\sum_{A\in G/(N\cdot\{\pm\bar{1}\})}\psi(A)f(A) \\ &= \det(f(AB^{-1}))_{A,B\in G/(N\cdot\{\pm\bar{1}\})} \\ &= \det(\omega(\bar{a}\bar{b}^{-1})\sum_{\bar{x}\in N}\delta(\bar{a}\bar{b}^{-1}\bar{x}))_{\bar{a},\bar{b}\in R} \\ &= \det(\sum_{\bar{x}\in N}\delta(\bar{a}\bar{b}^{-1}\bar{x}))_{\bar{a},\bar{b}\in R} \\ &= \det(C(\bar{a}\bar{b}^{-1}) - C'(\bar{a}\bar{b}^{-1}))_{\bar{a},\bar{b}\in R} \\ &= \pm\det(C(\bar{a}\bar{b}) - C'(\bar{a}\bar{b}))_{\bar{a},\bar{b}\in R}.\end{aligned}$$

From

$$C(\bar{a}\bar{b}) - C'(\bar{a}\bar{b}) = 2C(\bar{a}\bar{b}) - |N|$$

and

$$C(\bar{b}) - C'(\bar{b}) = |N| - 2C'(\bar{b}),$$

we obtain

$$\begin{aligned}\prod_{\chi\in X^-} \sum_{\bar{a}\in S} \chi(\bar{a}) &= \pm \det(2C(\bar{a}\bar{b}) - |N|)_{\bar{a},\bar{b}\in R} \\ &= \pm 2^{n-1}\det(C(\bar{a}\bar{b}) - C'(\bar{b}))_{\bar{a},\bar{b}\in R} \\ &= \pm 2^{n-1}\det D_{m,N}.\end{aligned}$$

This completes the proof of Theorem.

References

[1] S. Lang: Cyclotomic Fields. Springer-Verlag, Berlin, Hiderberg, New York (1978).

[2] J. W. Sands and W. Schwarz: A Demjanenko matrix for abelian fields of prime power conductor (preprint).

[3] W. Schwarz: Demjanenko matrix and 2-divisibility of class numbers. Arch. Math., **60**, 154–156 (1993).

[4] W. Sinnott: On the Stickelberger ideal and the circular units. Ann. of Math., **108**, 107–134 (1978).

[5] L. C. Washington: Introduction to Cyclotomic Fields. Springer-Verlag, Berlin, Hiderberg, New York (1982).