67. *Absolutely Continuous Spectra of Relativistic Schrödinger Operators with Magnetic Vector Potentials*

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§1. Introduction. A spinless particle in an electromagnetic field is described by a relativistic Schrödinger operator

$$H = h^w(x, D) + V(x),$$

where

$$h^w(x, D)u(x) = \iint e^{i(x-y)\cdot\xi}\, h\Big(\frac{x+y}{2}, \xi\Big)u(y)\,(2\pi)^{-n}dyd\xi,$$

$$h(x, \xi) = \sqrt{|\xi - a(x)|^2 + m^2},$$

$$(x, \xi \in \boldsymbol{R}^n,\ m > 0).$$

Although many efforts have been made to understand the nature of the operator H, there are few works in which spectral properties of H are investigated (cf. [2]-[7]). Indeed, Nagase and Umeda [8] is the only work locating the spectrum of the operator H as far as we know. In [8] they showed that $\sigma_{\text{ess}}(H) = [m, \infty)$ under the following assumptions (precisely speaking, their assumptions on $a(x)$ are weaker than (I) and (II) below):

(I) For any multi-index α, $\partial_x^\alpha a(x) \to 0$ as $|x| \to \infty$.

(II) $V(-\Delta + 1)^{-1/2}$ is a compact operator in $L^2(\boldsymbol{R}^n)$.

From the view point of spectral and scattering theory, it is natural to ask whether the absolutely continuous spectrum of H coincides with the interval $[m, \infty)$, and whether the singular continuous spectrum is empty. The aim of this note is to make a remark that the Enss theory can give answers to these questions. Finally, we mention here that our assumption on $a(x)$ is stronger than (I) and (II) above for technical reason.

§2. Results. Throughout this section, ε denotes a positive constant and $\langle x\rangle = \sqrt{1 + |x|^2}$. For the vector and scalar potentials $a(x) = (a_1(x), \dots, a_n(x))$ and $V(x)$, we make the following assumptions respectively:

(A) Each $a_j(x)$ is a C^∞-function such that $|\partial^\alpha a_j(x)| \le C_\alpha \langle x\rangle^{-1-\varepsilon}$ for any α.

(V) $V(x)$ is a real-valued measurable function such that $|V(x)| \le C\langle x\rangle^{-1-\varepsilon}$.

It is known [8, section 2] that under assumptions (A) and (V) the operator $h^w(x, D) + V(x)$ restricted on $C_0^\infty(\boldsymbol{R}^n)$ is essentially self-adjoint in $L^2(\boldsymbol{R}^n)$. Its self-adjoint realization will be denoted by H again. It is also known [8, section 2] that $\text{Dom}(H) = H^1(\boldsymbol{R}^n)$, the Sobolev space of order 1. Our result is

Theorem. *Let (A) and (V) be satisfied. Then*

(i) *H has no singular continuous spectrum. The absolutely continuous spectrum of H is the interval $[m, \infty)$.*

(ii) *m is the only possible limit point for the point spectrum of H. Any eigenvalue in the interval (m, ∞) has finite multiplicity.*

Remark. In the proof of the theorem above, we apply a theorem in [11], which is an extension of Enss [1]. See also Perry [9] and Simon [11], which are also extensions of [1].

Sketch of the proof. Let H_0 be the self-adjoint realization in $L^2(\boldsymbol{R}^n)$ of the operator $\sqrt{-\Delta + m^2}$ restricted on $C_0^\infty(\boldsymbol{R}^n)$. It is evident that $\mathrm{Dom}(H_0) = H^1(\boldsymbol{R}^n)$. Define $r(x, \xi) = h(x, \xi) - \sqrt{|\xi|^2 + m^2}$. It is straightforward that for every pair of multi-indices α and β

$$|\partial_\xi^\alpha \partial_x^\beta r(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-1-\varepsilon} \langle \xi \rangle^{-|\alpha|}. \tag{2.1}$$

In order to apply [11, Theorem 1], we need to rewrite the operator $r^w(x, D)$ as an operator with ordinary symbol: Defining

$$r^o(x, \xi) := \mathrm{Os} - \iint e^{-iy\cdot\eta}\, r\Big(x + \frac{y}{2}, \xi + \eta\Big)(2\pi)^{-n} dy d\eta,$$

we see that $r^w(x, D) = r^o(x, D)$. By integration by parts we deduce that $r^o(x, \xi)$ obeys the same inequality as in (2.1). We can now apply [11, Theorem 1] with $p(\xi) = \sqrt{|\xi|^2 + m^2}$, $a(x, \xi) = r^o(x, \xi) + V(x)$ and get all the conclusions of Theorem 1 of [11], which imply conclusions (i) and (ii) of the theorem.

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