65. A Constructive Approach to the Law Equivalence of Infinitely Divisible Random Measures

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We are concerned with our method to construct infinitely divisible random measures on $\boldsymbol{T}$ based on Poisson random measures on $\boldsymbol{S} = \boldsymbol{T} \times (\boldsymbol{R} \setminus \{0\})$. As an application we discuss the equivalence problem for infinitely divisible random measures on $\boldsymbol{T}$.

§1. Preliminaries. Let $\boldsymbol{T}$ be an arbitrary nonempty set and $\mathcal{T}$ be a δ-ring of subsets of $\boldsymbol{T}$. We assume there exists an increasing sequence $\{\boldsymbol{T}_n ; n \geq 1\} \subset \mathcal{T}$ with $\boldsymbol{T} = \cup_{n=1}^{\infty} \boldsymbol{T}_n$ and $\{t\} \in \mathcal{T}$ for each $t \in \boldsymbol{T}$. Let $\boldsymbol{\Lambda} = \{\Lambda(A) ; A \in \mathcal{T}\}$ be an *infinitely divisible random measure* (or **ID** *random measure*) on $\boldsymbol{T}$ with no Gaussian component, which is defined on a basic probability space $(\Omega, \mathcal{F}, \boldsymbol{P})$ (see [3]). In other words, $\boldsymbol{\Lambda}$ is a real stochastic process characterized by

$$(1.1) \qquad \boldsymbol{E}[\exp(iz\Lambda(A))] = \exp\left[izv(A) + \int\int_{A \times \boldsymbol{R}_0} g(z, x) M(dtdx)\right] \qquad (z \in \boldsymbol{R}, A \in \mathcal{T}),$$

where $g(z, x) = \exp(izx) - 1 - izx\mathbf{1}_J(x)$, $J = (-1,1)$ and $\boldsymbol{R}_0 = \boldsymbol{R} \setminus \{0\}$. Here v is an $\boldsymbol{R}$-valued signed measure on $\mathcal{T}$ and M is a measure on $\boldsymbol{S} = T \times \boldsymbol{R}_0$ satisfying

$$(1.2) \qquad \int\int_{A \times \boldsymbol{R}_0} (1 \wedge x^2) M(dtdx) < \infty \quad (A \in \mathcal{T}).$$

We mean by $\boldsymbol{\Lambda} =^d [v, M]$ that the probability law of $\boldsymbol{\Lambda}$ is determined by parameters v and M. We denote by $\boldsymbol{P}_\Lambda$ the probability measure on a measurable space $(\boldsymbol{R}^{\mathcal{T}}, \mathcal{F}(\boldsymbol{R}^{\mathcal{T}}))$ induced by the map $\Lambda : \Omega \ni \omega \rightarrow \Lambda(\cdot, \omega) \in \boldsymbol{R}^{\mathcal{T}}$, where $\boldsymbol{R}^{\mathcal{T}}$ is the set of all $\boldsymbol{R}$-valued functions on $\mathcal{T}$ and $\mathcal{F}(\boldsymbol{R}^{\mathcal{T}})$ is the σ-algebra on $\boldsymbol{R}^{\mathcal{T}}$ generated by all coordinate functions. The product measurable space $(\boldsymbol{S}, \mathcal{S})$ is given by $\mathcal{S} = \sigma(\mathcal{T}) \otimes \mathcal{B}(\boldsymbol{R}_0)$, where $\sigma(\mathcal{T})$ is the σ-algebra on $\boldsymbol{T}$ generated by $\mathcal{T}$ and $\mathcal{B}(\boldsymbol{R}_0)$ is the Borel σ-algebra on $\boldsymbol{R}_0$. Let $\mathcal{N} = \mathcal{N}(\boldsymbol{S})$ be the totality of nonnegative (possibly infinite) integer-valued measures on $(\boldsymbol{S}, \mathcal{S})$. Let $\mathcal{F}^+(\boldsymbol{S})$ be the set of all nonnegative measurable functions on $(\boldsymbol{S}, \mathcal{S})$. We denote by $\mathcal{B}(\mathcal{N})$ the σ-algebra on $\mathcal{N}$ generated by all functions f^* on $\mathcal{N}$ given by

$$f^*(\nu) = \langle \nu, f \rangle = \int_{\boldsymbol{S}} f d\nu \quad \text{for} \quad f \in \mathcal{F}^+(\boldsymbol{S}) \quad \text{and} \quad \nu \in \mathcal{N}.$$

An $\mathcal{N}$-valued random element ξ is called a *Poisson random measure* on $\boldsymbol{S}$ with intensity M if it is defined on $(\Omega, \mathcal{F}, \boldsymbol{P})$ and its Laplace transform is given by

(1.3) $$\boldsymbol{E}[\exp(-\langle\xi, f\rangle)] = \exp\Big[-\int\int_{\boldsymbol{S}}\{1-\exp(-f(t, x))\}M(dtdx)\Big]$$
for $f \in \mathscr{F}^+(\boldsymbol{S})$.

§2. A construction of infinitely divisible random measures. In this section we shall construct a version of $\boldsymbol{\Lambda} =^d [v, M]$ based on a Poisson random measure on $\boldsymbol{S}$. For simplicity we may assume $M(\boldsymbol{S}) > 0$.

Case (I): $M(\boldsymbol{S}) < \infty$. For each $k \geq 1$, let $(\boldsymbol{S}^k, \mathscr{S}^k, \boldsymbol{P}_k)$ be a probability space given by $\boldsymbol{P}_k = M(\boldsymbol{S})^{-k}M^k$, where we mean by $(\boldsymbol{S}^k, \mathscr{S}^k, M^k)$ the k-fold product measure space of $(\boldsymbol{S}, \mathscr{S}, M)$. Then we consider a probability space $(\Omega^*, \mathscr{F}^*, \boldsymbol{P}^*)$ defined by

(2.1) $$\Omega^* = \cup_{k=0}^{\infty}\boldsymbol{S}^k, \quad \mathscr{F}^* = \{A^* = \cup_{k=0}^{\infty}A_k\,;\, A_k \in \mathscr{S}^k\ (k \geq 0)\},$$
$$\boldsymbol{P}^*(A^*) = \exp(-M(\boldsymbol{S}))\sum_{k=0}^{\infty}(k!)^{-1}M(\boldsymbol{S})^k\boldsymbol{P}_k(A_k) \text{ for } A^* = \bigcup_{k=0}^{\infty}A_k \in \mathscr{F}^*,$$

where $(\boldsymbol{S}^0, \mathscr{S}^0, \boldsymbol{P}_0)$ is the trivial probability space given by $\boldsymbol{S}^0 = \{0\}$ and $\mathscr{S}^0 = \{\emptyset, \boldsymbol{S}^0\}$. We call $(\Omega^*, \mathscr{F}^*, \boldsymbol{P}^*)$ the *basic canonical probability space* associated with $(\boldsymbol{S}, \mathscr{S}, M)$. Let $\Phi : \Omega^* \to \mathscr{N}$ be an $\mathscr{F}^*/\mathscr{B}(\mathscr{N})$-measurable map given by $\Phi(0) = 0$ and

(2.2) $$\langle\Phi(\omega^*), f\rangle = \sum_{\iota=1}^{k} f(p_\iota(\omega^*)) \quad \text{for} \quad f \in \mathscr{F}^+(\boldsymbol{S})$$

when $\omega^* = (p_1(\omega^*), \cdots, p_k(\omega^*)) \in \boldsymbol{S}^k (k \geq 1)$. Then we obtain a Poisson random measure Φ on $\boldsymbol{S}$ with intensity M with respect to $\boldsymbol{P}^*$. We define

(2.3) $$\Lambda^*(A, \omega^*) = v(A) + \int\int_{A\times\boldsymbol{R}_0} x\Phi(dtdx, \omega^*) - \int\int_{A\times J} xM(dtdx)$$
$$(A \in \mathscr{T}, \omega^* \in \Omega^*),$$

where we put $\Phi(U, \omega^*) = [\Phi(\omega^*)](U)$ for $U \in \mathscr{S}$ and $\omega^* \in \Omega^*$. Then we have

Proposition 1. *The process $\boldsymbol{\Lambda}^* = \{\Lambda^*(A)\,;\, A \in \mathscr{T}\}$ is an* **ID** *random measure on $\boldsymbol{T}$ which is defined on $(\Omega^*, \mathscr{F}^*, \boldsymbol{P}^*)$ and characterized by $\boldsymbol{\Lambda}^* =^d [v, M]$.*

Case (II): $M(\boldsymbol{S}) = \infty$. On account of (1.2) we can choose a sequence $\{\boldsymbol{S}_n\,;\, n \geq 1\} \subset \mathscr{S}$ of disjoint subsets of $\boldsymbol{S}$ satisfying $\boldsymbol{S} = \cup_{n=1}^{\infty}\boldsymbol{S}_n$ and $0 < M(\boldsymbol{S}_n) < \infty$ $(n \geq 1)$. Let $\{M_n\,;\, n \geq 1\}$ be a sequence of finite measures on $(\boldsymbol{S}, \mathscr{S})$ defined by $M_n(U) = M(U \cap \boldsymbol{S}_n)$ for $U \in \mathscr{S}$. Let us introduce an infinite product probability space

(2.4) $$(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\boldsymbol{P}}) = \Pi_{n=1}^{\infty}(\Omega^*, \mathscr{F}^*, \boldsymbol{P}_n^*),$$

where $(\Omega^*, \mathscr{F}^*, \boldsymbol{P}_n^*)$ is the basic canonical probability space associated with $(\boldsymbol{S}, \mathscr{S}, M_n)$. We call $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\boldsymbol{P}})$ the *canonical probability space* assoacited with decomposition $M = \sum_{n=1}^{\infty}M_n$ on $(\boldsymbol{S}, \mathscr{S})$. Then we have Poisson random measures $\Psi_n = \sum_{\iota=1}^{n}(\Phi\circ\pi_\iota)$ and $\Psi = \sum_{n=1}^{\infty}(\Phi\circ\pi_n)$ on $\boldsymbol{S}$ respectively with intensities $M_{(n)} = \sum_{\iota=1}^{n}M_\iota$ and M. Here π_n denotes the n-th projection map from $\tilde{\Omega} = (\Omega^*)^\infty$ onto Ω^*. Now we define, for each $n \geq 1$,

(2.5) $$\tilde{\Lambda}_n(A, \tilde{\omega}) = v(A) + \int\int_{A\times\boldsymbol{R}_0} x\Psi_n(dtdx, \tilde{\omega}) - \int\int_{A\times J} xM_{(n)}(dtdx)$$
$$(A \in \mathscr{T}, \tilde{\omega} \in \tilde{\Omega}).$$

On account of the Lévy's equivalence theorem on the convergence of series with independent summands, we can find a random variable $\tilde{\Lambda}_\infty(A)$ to which

$\{\tilde{\Lambda}_n(A)\}$ converges almost surely as $n \to \infty$. Thus we have

Proposition 2. *The process* $\tilde{\boldsymbol{\Lambda}}_\infty = \{\tilde{\Lambda}_\infty(A) ; A \in \mathcal{T}\}$ *is an* **ID** *random measure on* $\boldsymbol{T}$ *which is defined on* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\boldsymbol{P}})$ *and characterized by* $\tilde{\boldsymbol{\Lambda}}_\infty =^d [v, M]$.

In the rest of this section we are concerned with a realization of $\boldsymbol{\Lambda}$ based on the space $\mathcal{N} = \mathcal{N}(\boldsymbol{S})$. We mean by $(\mathcal{N}, \mathcal{B}(\mathcal{N}), \boldsymbol{Q}^M)$ a probability space given by

(2.6) $\boldsymbol{Q}^M = [\boldsymbol{P}^*]_\Phi$ in Case (I) and $\boldsymbol{Q}^M = [\tilde{\boldsymbol{P}}]_\Psi$ in Case (II),

where $[\boldsymbol{P}^*]_\Phi$ and $[\tilde{\boldsymbol{P}}]_\Psi$ stand for the images of $\boldsymbol{P}^*$ and $\tilde{\boldsymbol{P}}$ induced by Φ and Ψ respectively. Then the identity map I on $(\mathcal{N}, \mathcal{B}(\mathcal{N}), \boldsymbol{Q}^M)$ is considered as a Poisson random measure on $\boldsymbol{S}$ with intensity M. Furthermore we can realize $\boldsymbol{\Lambda} =^d [v, M]$ in the space of $\boldsymbol{R}$-valued signed measures on $\mathcal{T}$ whenever the following conditions are satisfied.

(2.7) For each $A \in \mathcal{T}$, there exists $n \geq 1$ such that $A \subset \boldsymbol{T}_n$;

(2.8) $$\int\int_{A\times J} |x| \, M(dtdx) \equiv m(A) < \infty \quad (A \in \mathcal{T}).$$

Let $\boldsymbol{H}^+ = \{H^+(A) ; A \in \mathcal{T}\}$, $\boldsymbol{H}^- = \{H^-(A) ; A \in \mathcal{T}\}$ and $\boldsymbol{H} = \{H(A) ; A \in \mathcal{T}\}$ be **ID** random measures on $\boldsymbol{T}$, which are defined on $(\mathcal{N}, \mathcal{B}(\mathcal{N}), \boldsymbol{Q}^M)$ and expressed as follows:

(2.9) $$H^\pm(A, \nu) = v^\pm(A) + m(A) + \int\int_{A\times \boldsymbol{R}_0} x^\pm \nu(dtdx) - \int\int_{A\times J} x^\pm M(dtdx),$$

(2.10) $$H(A, \nu) = v(A) + \int\int_{A\times \boldsymbol{R}_0} x\nu(dtdx) - \int\int_{A\times J} xM(dtdx).$$

Here $v = v^+ - v^-$ stands for the Jordan decomposition of v. We put $\boldsymbol{R}_\pm = \{\pm x > 0\}$ and $M_\pm(U) = M(U \cap (\boldsymbol{T} \times \boldsymbol{R}_\pm))$ for $U \in \mathcal{S}$ respectively. Then we have

Theorem 1. *Assume* (2.7) *and* (2.8). *Then* $\boldsymbol{H}^\pm$ *and* $\boldsymbol{H}$ *are characterized by* $\boldsymbol{H}^\pm =^d [v^\pm + m, M_\pm]$ *and* $\boldsymbol{H} =^d [v, M]$. *Furthermore* $\boldsymbol{H}^+$ *and* $\boldsymbol{H}^-$ *are independent and also there exists a set* $\mathcal{N}_0 \in \mathcal{B}(\mathcal{N})$ *with* $\boldsymbol{Q}^M(\mathcal{N}_0) = 1$ *satisfying*

(2.11) $H(A, \nu) = H^+(A, \nu) - H^-(A, \nu)$ *and*

$$0 \leq H^\pm(A, \nu) < \infty \quad (A \in \mathcal{T}, \nu \in \mathcal{N}_0).$$

§3. The law equivalence of infinitely divisible random measures. In what follows we discuss the equivalence problem for **ID** random measures on $\boldsymbol{T}$ based on the method stated in Section 2. Given σ-finite measures μ and ν on a measurable space $(\boldsymbol{E}, \mathcal{E})$, we mean by $\mu \sim \nu$ that μ and ν are equivalent, i.e., mutually absolutely continuous. The Hellinger-Kakutani distance and inner product are defined respectively by

$$\mathrm{dist}(\mu, \nu) = \left[\int_{\boldsymbol{E}} (\sqrt{d\mu} - \sqrt{d\nu})^2\right]^{1/2} \quad \text{and} \quad \rho(\mu, \nu) = \int_{\boldsymbol{E}} \sqrt{d\mu d\nu}.$$

Theorem 2. *Let* $\boldsymbol{\Lambda}_1$ *and* $\boldsymbol{\Lambda}_2$ *be* **ID** *random measures on* $\boldsymbol{T}$ *given by* $\boldsymbol{\Lambda}_j =^d [v_j, M^{(j)}]$ $(j = 1, 2)$. *Then* $\boldsymbol{P}_{\Lambda_1} \sim \boldsymbol{P}_{\Lambda_2}$ *if the following three conditions hold simultaneously*:

(E.1) $M^{(1)} \sim M^{(2)}$,

(E.2) $\mathrm{dist}(M^{(1)}, M^{(2)}) < \infty$,

(E.3) $$v_1(A) - v_2(A) = \int\int_{A\times J} x\{M^{(1)} - M^{(2)}\}(dtdx) \quad (A \in \mathcal{T}).$$

§4. The outline of the proof of Theorem 2. First we construct versions of $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ based on Poisson random measures on $\boldsymbol{S}$ along the procedure stated in Section 2. For simplicity we devote ourselves to the case that $M^{(j)}(S) = \infty$ ($j = 1, 2$). Then we can find a decomposition $\boldsymbol{S} = \cup_{n=1}^{\infty} \boldsymbol{S}_n$ with $0 < M^{(j)}(S_n) < \infty (n \geq 1, j = 1, 2)$. For each $j = 1, 2$, we construct the canonical probability space

$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\boldsymbol{P}}^{(j)}) = \prod_{n=1}^{\infty} (\Omega^*, \mathcal{F}^*, \boldsymbol{P}_n^{*(j)})$$

associated with decomposition $M^{(j)} = \sum_{n=1}^{\infty} M_n^{(j)}$ on $(\boldsymbol{S}, \mathcal{S})$, where we put $M_n^{(j)}(U) = M^{(j)}(U \cap \boldsymbol{S}_n)$ for $U \in \mathcal{S}$. Now (E.1) implies $M_n^{(1)} \sim M_n^{(2)}$ and also $\boldsymbol{P}_n^{*(1)} \sim \boldsymbol{P}_n^{*(2)}$ for each $n \geq 1$. Further (E.2) implies

(4.1) $\quad \Pi_{n=1}^{\infty} \varrho(\boldsymbol{P}_n^{*(1)}, \boldsymbol{P}_n^{*(2)}) = \exp[-(1/2)\mathrm{dist}(M^{(1)}, M^{(2)})^2] > 0.$

Therefore we obtain $\tilde{\boldsymbol{P}}^{(1)} \sim \tilde{\boldsymbol{P}}^{(2)}$ by the Kakutani's theorem on the equivalence of infinite product probability measures (see [2]). By applying Proposition 2, we obtain stochastic processes $\tilde{\boldsymbol{\Lambda}}_{\infty}^{(j)} = \{\tilde{\Lambda}_{\infty}^{(j)}(A)\,;A \in \mathcal{T}\}$ ($j = 1, 2$) satisfying the following two conditions.

(4.2) $\tilde{\boldsymbol{\Lambda}}_{\infty}^{(j)}$ is defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\boldsymbol{P}}^{(j)})$ and characterized by $\tilde{\boldsymbol{\Lambda}}_{\infty}^{(j)} =^d [v_j, M^{(j)}]$;

(4.3) For each $A \in \mathcal{T}$, the sequence $\{\tilde{\Lambda}_n^{(j)}(A)\,; n \geq 1\}$ converges almost surely to $\tilde{\Lambda}_{\infty}^{(j)}(A)$ with respect to $\tilde{\boldsymbol{P}}^{(j)}$ as $n \to \infty$, where we put $M_{(n)}^{(j)} = \sum_{\ell=1}^{n} M_{\ell}^{(j)}$ and

(4.4) $$\tilde{\Lambda}_n^{(j)}(A, \tilde{\omega}) = v_j(A) + \int\int_{A \times \boldsymbol{R}_0} x\Psi_n(dtdx, \tilde{\omega}) - \int\int_{A \times J} xM_{(n)}^{(j)}(dtdx) \quad (A \in \mathcal{T}, \tilde{\omega} \in \tilde{\Omega}).$$

On account of (E.2) and (4.4) we have the following equations:

(4.5) $$\lim_{n\to\infty}\int\int_{A \times J} x\{M_{(n)}^{(1)} - M_{(n)}^{(2)}\}(dtdx) = \int\int_{A \times J} x\{M^{(1)} - M^{(2)}\}(dtdx),$$

(4.6) $$\tilde{\Lambda}_n^{(1)}(A, \tilde{\omega}) - \tilde{\Lambda}_n^{(2)}(A, \tilde{\omega}) = v_1(A) - v_2(A) - \int\int_{A \times J} x\{M_{(n)}^{(1)} - M_{(n)}^{(2)}\}(dtdx)$$

for $A \in \mathcal{T}$, $\tilde{\omega} \in \tilde{\Omega}$, and $n \geq 1$. Therefore combining (E.3) with $\tilde{\boldsymbol{P}}^{(1)} \sim \tilde{\boldsymbol{P}}^{(2)}$ yields that $\tilde{\Lambda}_{\infty}^{(1)}(A) = \tilde{\Lambda}_{\infty}^{(2)}(A)$ a.s. with respect to $\tilde{\boldsymbol{P}}^{(1)}$ and also $\tilde{\boldsymbol{P}}^{(2)}$. Now putting $\Theta(A, \tilde{\omega}) = \tilde{\Lambda}_{\infty}^{(1)}(A, \tilde{\omega})$ for $A \in \mathcal{T}$ and $\tilde{\omega} \in \tilde{\Omega}$, we have a process $\boldsymbol{\Theta} = \{\Theta(A)\,; A \in \mathcal{T}\}$ which is characterized by

(4.7) $\quad \boldsymbol{\Theta} =^d [v_j, M^{(j)}]$ with respect to $\tilde{\boldsymbol{P}}^{(j)}$ $(j = 1, 2)$.

This implies the equalities $\boldsymbol{P}_{\Lambda_j} = [\tilde{\boldsymbol{P}}^{(j)}]_{\Theta}$ ($j = 1, 2$), where $[\tilde{\boldsymbol{P}}^{(j)}]_{\Theta}$ stands for the image of $\tilde{\boldsymbol{P}}^{(j)}$ induced by the map $\Theta : \tilde{\Omega} \ni \tilde{\omega} \to \Theta(\cdot, \tilde{\omega}) \in \boldsymbol{R}^{\mathcal{T}}$. Thus we obtain the desired relation $\boldsymbol{P}_{\Lambda_1} \sim \boldsymbol{P}_{\Lambda_2}$ from $\tilde{\boldsymbol{P}}^{(1)} \sim \tilde{\boldsymbol{P}}^{(2)}$.

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