63. An Asymptotic Formula for the Kolmogorov Diffusion and a Refinement of Sinai's Estimates for the Integral of Brownian Motion

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1. Introduction and the main theorem. In connection with a problem of the inviscid Burgers equation ([10], [11]), Sinai ([12]) obtained the following estimates (1.3) and (1.4) for the distribution of the integral of one-dimensional Brownian motion $b(t)$ starting at 0: For $r > 0$, $a \in \boldsymbol{R}$, $A > 0$ and $\sigma > 0$, let

$$P_{ra}(A) = P\left\{\int_0^t b(u)\,du < at \text{ for all } 0 \leq t \leq A\right\} \tag{1.1}$$

and

$$P_{ra\sigma} = P\left\{\int_0^t b(u)\,du < r + at + \sigma t^2 \text{ for all } 0 \leq t \leq \infty\right\}. \tag{1.2}$$

Then, for each fixed $r > 0$ and $a \in \boldsymbol{R}$,

$$P_{ra}(A) \asymp A^{-1/4} \text{ as } A \uparrow \infty \tag{1.3}$$

and

$$P_{ra\sigma} \asymp \sigma^{1/2} \text{ as } \sigma \downarrow 0. \tag{1.4}$$

Here we use the following general notation; for positive functions $f(x)$ and $g(x)$ on $(0, \infty)$,

$$f(x) \asymp g(x) \text{ as } x \uparrow \infty[\downarrow 0] \text{ if } 0 < \liminf_{x \uparrow \infty[\downarrow 0]} \frac{g(x)}{f(x)} \leq \limsup_{x \uparrow \infty[\downarrow 0]} \frac{g(x)}{f(x)} < \infty$$

and

$$f(x) \sim g(x) \text{ as } x \uparrow \infty[\downarrow 0] \text{ if } \lim_{x \uparrow \infty[\downarrow 0]} \frac{g(x)}{f(x)} = 1.$$

In this paper, we refine these results in the following way:

$$P_{ra}(A) \sim C_1(r, a) A^{-1/4} \text{ as } A \uparrow \infty \tag{1.5}$$

and

$$P_{ra\sigma} \sim C_2(r, a) \sigma^{1/2} \text{ as } \sigma \downarrow 0. \tag{1.6}$$

Also we compute explicit expressions of $C_1(r, a)$ and $C_2(r, a)$. Our approach to this problem is based on an observation of a two dimensional diffusion process $(X(t), Y(t))$ defined by

$$Y(t) = y + b(t), \quad X(t) = x + \int_0^t Y(u)\,du = x + yt + \int_0^t b(u)\,du. \tag{1.7}$$

This diffusion is often called the *Kolmogorov diffusion* since it has been introduced by Kolmogorov [4]. As usual, we denote the probability law for the diffusion starting at $(x, y) \in \boldsymbol{R}^2$ by $P_{(x,y)}$ and let T be the first hitting time to the positive y-axis:

$$T = \inf\{t \geq 0\,;\, X(t) = 0,\ Y(t) \geq 0\}. \tag{1.8}$$

Then both asymptotics (1.5) and (1.6) can be obtained systematically from the following theorem.

Theorem. *For $a \geq 0$, $b \geq 0$ and $(x, y) \in \boldsymbol{R}^2$ with $x \leq 0$,*

(1.9) $1 - E_{(x,y)}(\exp[-a\sigma^2 T - b\sigma Y(T)]) \sim C(a, b; x, y)\sqrt{\sigma}$ *as* $\sigma \downarrow 0$.

The constant $C(a, b; x, y)$ is obviously zero if $a = b = 0$ or $y \geq 0$ and $x = 0$. Otherwise it is strictly positive and can be expressed in the following form

$$C(a, b; -\xi, -\eta) = \frac{3(b+\sqrt{2a})}{\sqrt{\pi}(b+2\sqrt{2a})\Gamma(1/6)}\int_0^\infty e^{-t}\left(\frac{9}{2}\xi t+\eta^3\right)^{1/6} t^{-5/6}dt, \quad \xi \geq 0,\ \eta > 0. \tag{1.10}$$

$$C(a, b; -\xi, \eta) = \frac{3(b+\sqrt{2a})}{\sqrt{\pi}(b+2\sqrt{2a})\Gamma(1/6)} e^{-\frac{2}{9}\frac{\eta^3}{\xi}} \times \int_0^\infty e^{-t}\left(\frac{9}{2}\xi t\right)^{1/6}\left(t+\frac{2}{9}\frac{\eta^3}{\xi}\right)^{-5/6} dt, \quad \xi > 0,\ \eta \geq 0. \tag{1.11}$$

As a corollary of this theorem, we can obtain (1.5) and (1.6) with

$$C_1(r, a) = \frac{1}{\Gamma\left(\frac{3}{4}\right)} C(1,0; -r, -a) \tag{1.12}$$

and

$$C_2(r, a) = C(2,2; -r, -a). \tag{1.13}$$

The proof of the theorem we give in section 2 depends heavily on a formula of McKean [8] for the joint distribution of T and $Y(T)$ with respect to the law $P_{(0,-1)}$ cf. also related results in [2], [3], [5]-[7]. Lachal [5] in fact obtained an expression for the joint distribution of $(T, Y(T))$ with respect to $P_{(x,y)}$ for any (x, y), but it seems difficult to derive (1.10) and (1.11) from his result. The reduction of (1.5) and (1.6) to the theorem will be given in section 3.

Finally we would thank Professor S. Kotani for helpful discussions and his suggestion to consider the Laplace transform v in section 2.

2. Proof of the theorem. The proof proceeds as the starting point (x, y) being situated first in the negative y-axis, then in the negative x-axis, in the third quadrant and finally in the second quadrant.

2.1. Preliminaries. For $a \geq 0$, $b \geq 0$ and $\sigma > 0$, let u be the function on the left half plane $\{(x, y) \mid x \leq 0\}$ of $\boldsymbol{R}^2$ defined by

$$u(x, y)(= u(x, y; \sigma, a, b)) = E_{(x,y)}\{\exp[-a\sigma^2 T - b\sigma Y(T)]\}. \tag{2.1}$$

Then u is uniquely determined by the following properties: $0 < u < 1$ and satisfies in the half plane $\{x < 0\}$ the equation

$$Lu := \frac{1}{2}\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial x} = a\sigma^2 u \tag{2.2}$$

with the boundary condition on the positive y-axis:

$$\lim_{x \uparrow 0} u(x, y) = e^{-b\sigma y}, \quad y > 0. \tag{2.3}$$

By the scaling property of the Brownian motion, we deduce easily that

$$u(x, y; c\sigma, a, b) = u(c^3 x, cy; \sigma, a, b), \quad c > 0. \tag{2.4}$$

Let

$$v(y; \sigma, \beta)(= v(y; \sigma, \beta, a, b)) = \int_{-\infty}^0 e^{\beta x} u(x, y; \sigma, a, b)dx, \quad \beta > 0. \tag{2.5}$$

By an integration by parts, we see that $v(y) = v(y\,;\sigma,\beta)$ satisfies for $y > 0$ the equation

$$(2.6)\qquad \frac{1}{2}v''(y) = (\beta y + a\sigma^2)v(y) - ye^{-b\sigma y}.$$

Let $\phi(y)$, $\psi(y)$, $F(y)$ be defined by

$$\phi(y) = \Gamma\Big(\frac{2}{3}\Big)3^{-1/3}\sqrt{y}\,I_{-\frac{1}{3}}\Big(\frac{2}{3}y^{3/2}\Big),\ \psi(y) = \Gamma\Big(\frac{1}{3}\Big)3^{-2/3}\sqrt{y}\,I_{\frac{1}{3}}\Big(\frac{2}{3}y^{3/2}\Big)$$

$$F(y) = \phi(y) - \frac{3^{1/3}\Gamma(2/3)}{\Gamma(1/3)}\psi(y) = \frac{2\cdot 3^{-1/3}}{\Gamma(1/3)}\sqrt{y}\,K_{\frac{1}{3}}(2/3\,y^{3/2}).$$

This $F(y)$ is a constant multiple of the Airy function $Ai(y)$. We introduce

$$v_1(y) = F(\sqrt[3]{2\beta}y + s)/F(s),$$

$$v_2(y) = \frac{1}{\sqrt[3]{2\beta}}\{-\psi(s)\phi(\sqrt[3]{2\beta}y + s) + \phi(s)\psi(\sqrt[3]{2\beta}y + s)\},$$

where $s = \dfrac{2a\sigma^2}{(2\beta)^{2/3}}$. Then $v_1(y)$, $v_2(y)$ are solutions of $\frac{1}{2}v''(y) = (\beta y + a\sigma^2)v(y)$ such that $v_1(0) = 1$, $v_1(y)$ is bounded on $(0,\infty)$, and $v_2(0) = 0$, $v_2'(0) = 1$. Hence

$$(2.7)\quad v(y\,;\sigma,\beta) = 2v_1(y)\int_0^y v_2(\xi)\xi e^{-b\sigma\xi}d\xi + 2v_2(y)\int_y^\infty v_1(\xi)\xi e^{-b\sigma\xi}d\xi + v(0\,;\sigma,\beta)v_1(y).$$

Let

$$(2.8)\qquad T_0 = \inf\{t \geq 0\,; Y(t) = 0\},\quad X^0 = X(T_0).$$

It is obvious that $X^0 < 0$ a.s. $(P_{(0,-y)})$ if $y > 0$. Applying the optional sampling theorem to the local martingale

$$(2.9)\qquad F(-\sqrt[3]{2\lambda}\,Y(t))e^{\lambda\int_0^t Y(u)du},\quad \lambda > 0,$$

we obtain $E_{(0,-y)}[e^{\lambda X^0}] = F(\sqrt[3]{2\lambda}y)$.

We can easily invert the Laplace transform to obtain the following:

$$(2.10)\quad \frac{P_{(0,-y)}(-X^0 \in dx)}{dx} = \frac{2^{1/3}y}{3^{2/3}\Gamma\Big(\frac{1}{3}\Big)x^{4/3}}e^{-2y^3/9x},\quad x > 0,\ y > 0.$$

Also it is well-known that

$$(2.11)\qquad E_{(0,-y)}[e^{-\frac{\mu^2}{2}T_0}] = e^{-\mu y},\quad y > 0,\ \mu > 0.$$

2.2 The case of the starting point on the negative y-axis. McKean ([8]) obtained the following formula:

$$P_{(0,-1)}[T \in dt,\ Y(T) \in dh] = \frac{3h}{\pi\sqrt{2}t^2}e^{-\frac{2}{t}(1-h+h^2)}\int_0^{4h/t}\frac{e^{-\frac{3}{2}\theta}}{\sqrt{\pi\theta}}d\theta\cdot dh\,dt,\quad t, h > 0.$$

Then, as $\sigma \downarrow 0$, we have

$$1 - u(0,-1\,;\sigma,a,b) = E_{(0,-1)}[1 - e^{-a\sigma^2 T - b\sigma Y(T)}]$$

$$= \int_0^\infty\int_0^\infty dt\,dh(1 - e^{-a\sigma^2 t - b\sigma h})\frac{3h}{\sqrt{2}\pi t^2}e^{-\frac{2}{t}(1-h+h^2)}\int_0^{4h/t}\frac{e^{-\frac{3}{2}\theta}}{\sqrt{\pi\theta}}d\theta$$

$$= \sqrt{\sigma}\int_0^\infty\int_0^\infty dt\,dh\frac{(1 - e^{-t-h})}{\sqrt{2}\pi b^2t^2}e^{-\frac{2a}{b^2t}(h^2 - b\sigma h + b^2\sigma^2)}\int_0^{\frac{4ah}{bt}}\frac{e^{-\frac{3}{2}\sigma\theta}}{\sqrt{\pi\sigma\theta}}\sigma d\theta$$

$$\sim \sqrt{\sigma}\,\frac{12a^{3/2}}{\sqrt{2}\pi^{3/2}b^{5/2}}\int_0^\infty\int_0^\infty dt\,dh(1-e^{-t-h})\,\frac{h^{3/2}}{t^{5/2}}\,e^{-\frac{2ah^2}{b^2t}}$$

$$=\frac{3(b+\sqrt{2a})}{\sqrt{\pi}\sqrt{b+2\sqrt{2a}}}\sqrt{\sigma}$$

by noting that $\int_0^\infty t^{-5/2}e^{-2ah^2/b^2t}dt = \frac{b^3}{a^{3/2}h^3 2^{3/2}}\sqrt{\pi}/2$, $\int_0^\infty t^{-5/2}e^{-2ah^2/b^2t-t}dt = \left(\frac{\sqrt{2a}}{b}h\right)^{-3/2}\int_0^\infty \tau^{-5/2}e^{-\frac{\sqrt{2a}}{b}h\left(\tau+\frac{1}{\tau}\right)}d\tau = 2^{1/4}a^{-3/4}b^{3/2}h^{-3/2}K_{-3/2}\left(2\frac{\sqrt{2a}}{b}h\right)$, and $K_{-\frac{3}{2}}(z) = \sqrt{\frac{\pi}{2z}}\,e^{-z}\left(1+\frac{1}{z}\right)$.

This proves that (1.9) holds for $(x, y) = (0, -1)$ with

$$C(a, b; 0, -1) = \frac{3(b+\sqrt{2a})}{\sqrt{\pi(b+2\sqrt{2a})}} \tag{2.12}$$

and, by the scaling property (2.4), it is immediately seen that (1.9) holds on the negative y-axis with

$$C(a, b; 0, -y) = \frac{3(b+\sqrt{2a})\sqrt{y}}{\sqrt{\pi(b+2\sqrt{2a})}},\quad y>0. \tag{2.13}$$

2.3. The case of the starting point on the negative x-axis. By the strong Markov property applied to the hitting time T_0 in (2.8) it follows that, for $y>0$,

$$u(0, -y; \sigma, a, b) = E_{(0,-y)}[e^{-a\sigma^2T_0}u(X^0, 0; \sigma, a, b)]. \tag{2.14}$$

Then, noting (2.11) and the scaling property (2.4),

$$C(a,b;0,-y)\sqrt{\sigma} \sim 1-u(0,-y;\sigma,a,b) \sim E_{(0,-y)}[1-u(X^0,0;\sigma,a,b)] = E_{(0,-y)}[1-u(-1,0;|X^0|^{1/3}\sigma,a,b)] \text{ as } \sigma\downarrow 0$$

and, by (2.10), this is equal to

$$\frac{2^{1/3}y}{3^{2/3}\Gamma\left(\frac{1}{3}\right)}\int_0^\infty e^{-2y^3/9x}x^{-4/3}(1-u(-1,0;x^{1/3}\sigma,a,b))dx. \tag{2.15}$$

Now we can apply a Tauberian theorem to conclude that

$$1-u(-1,0;\sigma,a,b) \sim C(a,b;-1.0)\sqrt{\sigma},\quad \text{as } \sigma\downarrow 0 \tag{2.16}$$

and the constant $C(a, b; -1,0)$ satisfies

$$C(a,b;-1,0)\frac{2^{1/3}y}{3^{2/3}\Gamma\left(\frac{1}{3}\right)}\int_0^\infty e^{-2y^3/9x}x^{-4/3}x^{1/6}dx = C(a,b;0,-y) = C(a,b;0,-1)\sqrt{y}.$$

By the scaling property (2.4),

$$1-u(-x,0;\sigma,a,b) = 1-u(-1,0;x^{1/3}\sigma,a,b) \sim C(a,b;-1,0)\sqrt{\sigma}x^{1/6},\quad \text{as } \sigma\downarrow 0, \tag{2.17}$$

that is (1.9) holds on the negative x-axis with

$$C(a,b;-x,0) = \frac{\Gamma\left(\frac{1}{3}\right)3^{1/3}}{\Gamma\left(\frac{1}{6}\right)2^{1/6}}x^{1/6}C(a,b;0,-1). \tag{2.18}$$

2.4. The case of the starting point in the third quadrant. Clearly the joint distribution of $(T_0, X^0 = X(T_0))$ with respect to the law $P_{(-x,-y)}$ is that of $(T_0, -x+X^0)$ with respect to $P_{(0,-y)}$. Then, by the strong Markov property and (2.17) together with a careful estimate, we can show that

$$\begin{aligned}1-u(-x,-y;\sigma,a,b) &= 1-E_{(0,-y)}[e^{-a\sigma^2 T_0}u(-x+X^0,0;\sigma,a,b)]\\ &\sim E_{(0,-y)}[1-u(-x+X^0,0;\sigma,a,b)]\\ &\sim \sqrt{\sigma}\,C(a,b;-1,0)E_{(0,-y)}[(x-X^0)^{1/6}],\ x>0,\ y>0.\end{aligned}$$

This proves that (1.9) holds for $(-x,-y)$, $x>0$, $y>0$ with

$$\begin{aligned}C(a,b;-x,-y) &= C(a,b;-1,0)E_{(0,-y)}[(x-X^0)^{1/6}]\\ &= \frac{C(a,b;0,-1)}{\Gamma\left(\frac{1}{6}\right)}\int_0^\infty t^{-5/6}e^{-t}\left(\frac{9xt}{2}+y^3\right)^{1/6}dt.\end{aligned}$$

2.5. The case of the starting point in the second quadrant. Let $v(y;\sigma,\beta)$ be defined by (2.5). Then by the result in the subsection 2.3,

$$\begin{aligned}v(0;\sigma,\beta) &= \int_0^\infty u(-x,0;\sigma,a,b)e^{-\beta x}dx\\ &\sim \frac{1}{\beta}-\sqrt{\sigma}\beta^{-7/6}\Gamma\left(\frac{7}{6}\right)C(a,b;-1,0).\end{aligned}$$

Note that functions $\phi(y)$, $\psi(y)$ and $F(y)$ in the subsection 2.1 are entire functions. Then from (2.7), we see by Taylor expansions and integration by parts that as $\sigma\downarrow 0$ and $\beta>0$ being fixed,

$$v(y;\sigma,\beta)\sim\frac{1}{\beta}-\sqrt{\sigma}\beta^{-7/6}\Gamma\left(\frac{7}{6}\right)C(a,b;-1,0)F(\sqrt[3]{2\beta}y)\quad\text{as } \sigma\downarrow 0.$$

From this we can conclude by a standard argument that, for $x>0$ and $y>0$,

(2.19) $$1-u(-x,y;\sigma,a,b)\sim\sqrt{\sigma}C(a,b;-x,y)$$

with

$$\int_0^\infty C(a,b;-x,y)e^{-\beta x}dx=\beta^{-7/6}\Gamma\left(\frac{7}{6}\right)C(a,b;-1,0)F(\sqrt[3]{2\beta}y).$$

We can invert this (cf. Oberhettinger and Badii [9], (13.45)) to obtain

(2.20) $$C(a,b;-x,y)=C(a,b;-1,0)\frac{\Gamma\left(\frac{7}{6}\right)3^{2/3}}{2^{1/3}\Gamma\left(\frac{1}{3}\right)}\frac{\sqrt{x}}{y}e^{-\frac{y^3}{9x}}W_{-\frac{1}{2},\frac{1}{6}}\left(\frac{2y^3}{9x}\right)$$

where $W_{-\frac{1}{2},\frac{1}{6}}(\cdot)$ is the Whittaker function. Noting

(2.21) $$W_{-\frac{1}{2},\frac{1}{6}}(z)=\frac{e^{-\frac{z}{2}}z^{1/3}}{\Gamma\left(\frac{7}{6}\right)}\int_0^\infty e^{-t}\frac{t^{1/6}}{(z+t)^{5/6}}dt,$$

(cf. Abramowitz [1], (13.2.5) and (13.1.33)) and (2.18), we finally obtain

$$\begin{aligned}&C(a,b;-x,y)\\ &=\frac{C(a,b;0,-1)}{\Gamma\left(\frac{1}{6}\right)}e^{-\frac{2y^3}{9x}}\int_0^\infty e^{-t}\frac{\left(\frac{9xt}{2}\right)^{1/6}}{\left(\frac{2y^2}{9x}+t\right)^{5/6}}dt.\end{aligned}$$

3. The reduction of (1.5) and (1.6) to the theorem. Since $P_{ra}(A)=P_{(-r,-a)}(T>A)$, (1.5) with (1.12) is an immediate consequence of the Tauberian theorem applied to

(3.1) $$1-E_{(-r,-a)}[e^{-\sigma^2 T}]\sim C(1,0;-r,-a)\sqrt{\sigma}\ \text{as}\ \sigma\downarrow 0.$$

For $\sigma \geq 0$, we consider the *Kolmogorov diffusion with drift*:

$$Y(t) = y + b(t) - 2\sigma t,\ X(t) = x + \int_0^t Y(u)\,du \tag{3.2}$$
$$= x + yt - \sigma t^2 + \int_0^t b(u)\,du.$$

The probability law of this diffusion starting at (x, y) is denoted by $P^\sigma_{(x,y)}$ so that $P^0_{(x,y)} = P_{(x,y)}$. Then by Girsanov's theorem, we have for any $n > 0$,

$$\begin{aligned} P^\sigma_{(x,y)}[T < n] &= E_{(x,y)}[e^{-2\sigma[Y(n)-y]-\frac{(2\sigma)^2}{2}n};\ T < n] \\ &= E_{(x,y)}[e^{-2\sigma[Y(T\wedge n)-y]-\frac{(2\sigma)^2}{2}T\wedge n};\ T < n] \\ &= e^{2\sigma y} E_{(x,y)}[e^{-2\sigma Y(T)-\frac{(2\sigma)^2}{2}T};\ T < n]. \end{aligned}$$

Noting $P_{(x,y)}(T < \infty) = 1$, and lettig $n \to \infty$, we have

$$\begin{aligned} P_{ra\sigma} = P^\sigma_{(-r,-a)}(T = \infty) &= 1 - e^{-2\sigma a} E_{(-r,-a)}[e^{-2\sigma Y(T)-\frac{(2\sigma)^2}{2}T}] \\ &= 1 - E_{(-r,-a)}[e^{-2\sigma Y(T)-\frac{(2\sigma)^2}{2}T}] + O(\sigma), \quad \text{as } \sigma \downarrow 0. \end{aligned}$$

Hence by the theorem,

$$P_{ra\sigma} \sim \sqrt{\sigma}\, C(2,2;\, -r,\, -a) \quad \text{as } \sigma \downarrow 0 \tag{3.3}$$

and we obtain (1.6) with $C_2(r, a) = C(2,2;\, -r,\, -a)$.

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