62. *Subgraphs of W-graphs and the 3-parallel Version Polynomial Invariants of Links*

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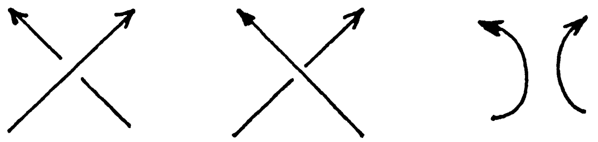
1. Introduction. The purpose of this paper is to give a method to compute the 3-papallel version of a special type of 2-variable Jones polynomials of 3 and 4 braids which can distinguish Kinoshita-Terasaka knot KT and Conway's 11-crossing knot KC. Generally speaking, it is necessary for a direct calculation of the 3-parallel version of a 2-variable Jones polynomial of a 4 braid of length r to compute about $10r$ many times products of matrixes of size n up to 7700. But our method needs the similar products of matrixes of size n up to only 98.

Freyd and Yetter, Lickorish and Millet, Ocneanu, and Hoste discovered in [1] a two-variable polynomial invariant $P_L(t, \chi)$ of an oriented link L.

Let L be an oriented link, and L_+, L_- and L_0 be links that have regular projections identical, except in one crossing where they are as in Fig. 1:



L_+ L_- L_0

Fig. 1

Then $P_L(t, \chi)$ is the Laurent polynomial defined by the following Conway relation:

(1) $P_L(t, \chi) = 1$, if L is the trivial knot.

(2) $t^{-1}P_{L+} - tP_{l_-} = \chi P_{L_0}$.

On the other hand, for an n-braid α in the braid group B_n, let

$$X_L(q, \lambda) = \left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1}(\sqrt{\lambda})^e tr(\pi(\alpha))$$

where L is the closed braid of α, e is the exponent sum of α and π is the representation of B_n in the Hecke algebra $H(q, n)$ sending the standard generators of B_n to those of $H(q, n)$. Then the invariant satisfies $P_L(t, \chi) = X_L(q, \lambda)$ if $t = \sqrt{\lambda}\sqrt{q}$ and $\chi = \left(\sqrt{q} - \frac{1}{\sqrt{q}}\right)$.

It is already known [6], [9] that no polynomial invariants of Conway type can distinguish two different mutant knots. But one of the author found

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in [8] that the 3-parallel version of 2-variable Jones polynomials distinguish certain mutant knots, and Morton and Traczyk also did in [9] that similar invariants distinguish the knots KT and KC by the direct calculation of the 3-parallel version of 2-variable Jones polynomials using Conway relation.

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2. 3-parallel version of polynomial invariants of links. Let L be a link, α be an element of the braid group B_n whose closure is isotopic to L, and β be the 3-parallel version of α (see Section 1.2 of [8]). Then the following Laurent polynomial

$$\chi_L^{(3)}(q,\lambda)=\left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1}(\sqrt{\lambda})^e tr(\pi(\beta))$$

is a polynomial invariant of L. Let $H(q, n)$ be the Iwahori-Hecke algebra of type A_{n-1} with the standard generators $g_1, g_2, \ldots, g_{n-1}$ and $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ be the standard generators of B_n. Let $\Psi_n^{(3)}: CB_n \to H(q, 3n)$ be the algebra homomorphism defined by

$$\Psi_n^{(3)}(\sigma_i)=g(3i-2,3i-1)^{-3}g(3i,3i+2)g(3i-1,3i+1)g(3i-2,3i) \qquad (1\le i\le n-1)$$

where $g(i, j)=g_i g_{i+1}\cdots g_j$. Then the computation of $\chi_L^{(3)}(q,\lambda)$ derives from computing the trace $tr(\Psi_n^{(3)}(\alpha))$.

In [10], the irreducible representations of $H(q, 9)$ and $H(q, 12)$ are given. Thus at present, we can compute $\chi_L^{(3)}(q,\lambda)$ for every link L whose braid form has braid index of 3 or 4. But no direct calculations of $\chi_L^{(3)}(q,\lambda)$ is well adapted for computer calculations as it involves product calculations of matrixes with very big size.

3. Certain subgraphs of W-graphs. Let $\Lambda(3n)$ be the set of partitions of a positive integer $3n$, Y be a Young diagram associated with $\Lambda(3n)$, $G(Y)$ be the W-graph with the vertex set $V(Y)=\{\chi_1, \chi_2, \ldots, \chi_s\}$ labelled by $I(G(Y))=\{I(\chi_1), I(\chi_2), \ldots, I(\chi_s)\}$ corresponding to Y, where $I(\chi_j)$ is the I-invariant of χ_j (see the definition in [3,4]). By making use of the results in [10], we get the list of $G(Y)$ for Young diagram Y associated with $\Lambda(9)$ and $\Lambda(12)$. We define a subset $V^3(Y)$ of $V(Y)$ as follows:

Each vertex x in $V(Y)$ is included in $V^3(Y)$ if and only if $I(x)$ contains all numbers k in $I=\{1, 2, \ldots, 3n\}$ with $k\equiv 2 \pmod 3$ and no numbers k in I with $k\equiv 1 \pmod 3$.

Let π_Y be the representation given by the W-graph $G(Y)$. Then, by using property of W-graphs, we know that the restriction to $\pi_Y\circ\Psi^{(3)}$ on the subspace U spanned by the basis corresponding to $V^3(Y)$ gives a reprsentation of B_n. Let $\omega_Y^{(3)}$ denote the trace of this representation, then $\omega_Y^{(3)}$ corresponds to $\omega_{\mu,\nu}$ in Theorem 1.4.10 of [8] where μ (resp. ν) corrresponds to the irreducible representation of $H(q, 3n)$ (resp. $H(q, 3)$) parametrized by Y (resp. (2.1)). Hence, by Theorem 1.5.1 of [8], the following $\chi_L^{(3)}(q,\lambda)^*$ is an invariant of knots.

$$\chi_L^{(3)}(q,\lambda)^*=\left(-\frac{1-\lambda q}{\sqrt{\lambda}(1-q)}\right)^{n-1}(\sqrt{\lambda})^e\sum_Y W_Y(q,z)\,\omega_Y^{(3)}(\alpha)$$

(see the definition of $W_Y(q, z)$ in [5]).

Let K and K' be two mutant knots. Then, by using Theorem 1.5.1 and 6.2.4 in [8], $X_K^{(3)}(q, \lambda) \neq X_{K'}^{(3)}(q, \lambda)$ if and only if $X_K^{(3)}(q, \lambda)^* \neq X_{K'}^{(3)}(q, \lambda)^*$. The maximal size of matrixes we need to compute $X_K^{(3)}(q, \lambda)^*$ is much less than that of $X_K^{(3)}(q, \lambda)$ and we can compute $X_K^{(3)}(q, \lambda)^*$ actually. To simplify the computation, we specialize $\lambda = q^r$ for a positive integer r. If $r = 1$ or 2, we can show that $X_K^{(3)}(q, \lambda)^* = X_{K'}^{(3)}(q, \lambda)^*$ for any mutant knots K and K'.

Next we tried to compute $X_L^{(3)}(q, q^3)^*$ for certain mutant knots:

KT (resp. KC) is the closure of a braid $\sigma_1\sigma_3^{-1}\sigma_2\sigma_3^{-1}\sigma_2^2\sigma_3^{-1}\sigma_1^4\sigma_2^2$ (resp. $\sigma_1\sigma_3^{-1}\sigma_2\sigma_3^{-1}\sigma_2^2\sigma_3^{-1}\sigma_1\sigma_2^{-3}$). We have the following result, where $x = q^2$.

$X_{KT}^{(3)}(q, q^3)^* = -x^{\wedge}78 + 2x^{\wedge}76 - 3x^{\wedge}74 + 6x^{\wedge}72 - 6x^{\wedge}70 + 3x^{\wedge}68$
$+ 4x^{\wedge}66 - 14x^{\wedge}64 + 32x^{\wedge}62 - 53x^{\wedge}60 + 68x^{\wedge}58 - 92x^{\wedge}56 + 112x^{\wedge}54$
$- 120x^{\wedge}52 + 108x^{\wedge}50 - 106x^{\wedge}48 + 97x^{\wedge}46 - 66x^{\wedge}44 + 19x^{\wedge}42$
$+ 35x^{\wedge}40 - 86x^{\wedge}38 + 172x^{\wedge}36 - 241x^{\wedge}34 + 313x^{\wedge}32 - 351x^{\wedge}30$
$+ 384x^{\wedge}28 - 368x^{\wedge}26 + 313x^{\wedge}24 - 258x^{\wedge}22 + 160x^{\wedge}20 - 60x^{\wedge}18$
$- 75x^{\wedge}16 + 169x^{\wedge}14 - 273x^{\wedge}12 + 356x^{\wedge}10 - 388x^{\wedge}8 + 399x^{\wedge}6$
$- 364x^{\wedge}4 + 328x^{\wedge}2 - 232 + 146x^{\wedge} - 2 - 73x^{\wedge} - 4 + 4x^{\wedge} - 6 + 54x^{\wedge}$
$- 8 - 107x^{\wedge} - 10 + 119x^{\wedge} - 12 - 131x^{\wedge} - 14 + 130x^{\wedge} - 16 - 119x^{\wedge}$
$- 18 + 100x^{\wedge} - 20 - 78x^{\wedge} - 22 + 60x^{\wedge} - 24 - 33x^{\wedge} - 26 + 16x^{\wedge} -$
$28 - 7x^{\wedge} - 30 - x^{\wedge} - 32 + 6x^{\wedge} - 34 + 6x^{\wedge} - 36 + 3x^{\wedge} - 38 - 2x^{\wedge}$
$- 40 + x^{\wedge} - 42$

$X_{KC}^{(3)}(q, q^3)^* = -x^{\wedge}78 + 2x^{\wedge}76 - 3x^{\wedge}74 + 6x^{\wedge}72 - 7x^{\wedge}70 + 8x^{\wedge}68$
$- 7x^{\wedge}66 + 3x^{\wedge}64 + 7x^{\wedge}62 - 19x^{\wedge}60 + 28x^{\wedge}58 - 50x^{\wedge}56 + 73x^{\wedge}54$
$- 87x^{\wedge}52 + 83x^{\wedge}50 - 93x^{\wedge}48 + 97x^{\wedge}46 - 79x^{\wedge}44 + 46x^{\wedge}42 - 8x^{\wedge}40$
$- 25x^{\wedge}38 + 96x^{\wedge}36 - 151x^{\wedge}34 + 211x^{\wedge}32 - 246x^{\wedge}30 + 283x^{\wedge}28$
$- 279x^{\wedge}26 + 242x^{\wedge}24 - 207x^{\wedge}22 + 134x^{\wedge}20 - 60x^{\wedge}18 - 49x^{\wedge}16$
$+ 118x^{\wedge}14 - 202x^{\wedge}12 + 267x^{\wedge}10 - 287x^{\wedge}8 + 294x^{\wedge}6 - 262x^{\wedge}4$
$+ 238x^{\wedge}2 - 156 + 85x^{\wedge} - 2 - 30x^{\wedge} - 4 - 23x^{\wedge} - 6 + 67x^{\wedge} - 8 -$
$107x^{\wedge} - 10 + 106x^{\wedge} - 12 - 106x^{\wedge} - 14 + 97x^{\wedge} - 16 - 80x^{\wedge} - 18 +$
$58x^{\wedge} - 20 - 38x^{\wedge} - 22 + 26x^{\wedge} - 24 - 8x^{\wedge} - 26 - x^{\wedge} - 28 - 4x^{\wedge} - 30$
$- 6x^{\wedge} - 32 + 7x^{\wedge} - 34 - 6x^{\wedge} - 36 + 3x^{\wedge} - 38 - 2x^{\wedge} - 40 + x^{\wedge} - 42$

Let K_1, K_2 be mutant knots such that K_1 (resp. K_2) has a braid form $\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_2^{-3}\sigma_1\sigma_2^{-2}\sigma_3^2\sigma_2\sigma_1\sigma_3^3\sigma_2^{-1}\sigma_3^2$ (resp. $\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_2^{-3}\sigma_1\sigma_2^3\sigma_3^2\sigma_2\sigma_1\sigma_3^{-2}\sigma_2^{-1}\sigma_3^2$).

$X_{K_1}^{(3)}(q, q^3)^* = x^{\wedge}94 - 2x^{\wedge}92 + 3x^{\wedge}90 - 6x^{\wedge}88 + 6x^{\wedge}86 - 4x^{\wedge}84$
$+ 8x^{\wedge}80 - 20x^{\wedge}78 + 33x^{\wedge}76 - 38x^{\wedge}74 + 51x^{\wedge}72 - 57x^{\wedge}70 + 50x^{\wedge}68$
$- 32x^{\wedge}66 + 14x^{\wedge}64 - 3x^{\wedge}62 - 38x^{\wedge}60 + 71x^{\wedge}58 - 104x^{\wedge}56 +$
$113x^{\wedge}54 - 126x^{\wedge}52 + 128x^{\wedge}50 - 95x^{\wedge}48 + 76x^{\wedge}46 - 24x^{\wedge}44 + x^{\wedge}42$
$+ 56x^{\wedge}40 - 66x^{\wedge}38 + 71x^{\wedge}36 - 89x^{\wedge}34 + 70x^{\wedge}32 - 79x^{\wedge}30 + 40x^{\wedge}28$
$- 60x^{\wedge}26 + 39x^{\wedge}24 - 42x^{\wedge}22 + 53x^{\wedge}20 - 47x^{\wedge}18 + 50x^{\wedge}16 - 19x^{\wedge}14$
$+ 30x^{\wedge}12 - 6x^{\wedge}10 - 14x^{\wedge}8 + 48x^{\wedge}6 - 59x^{\wedge}4 + 69x^{\wedge}2 - 81 + 77x^{\wedge}$
$- 2 - 64x^{\wedge} - 4 + 40x^{\wedge} - 6 - 21x^{\wedge} - 8 - 6x^{\wedge} - 10 + 22x^{\wedge} - 12$
$- 29x^{\wedge} - 14 + 29x^{\wedge} - 16 - 26x^{\wedge} - 18 + 20x^{\wedge} - 20 - 11x^{\wedge} - 22 +$
$6x^{\wedge} - 24 + 2x^{\wedge} - 26$

$X_{K_2}^{(3)}(q, q^3)^* = x^{\wedge}94 - 2x^{\wedge}92 + 3x^{\wedge}90 - 6x^{\wedge}88 + 6x^{\wedge}86 - 4x^{\wedge}84 +$
$8x^{\wedge}80 - 21x^{\wedge}78 + 38x^{\wedge}76 - 49x^{\wedge}74 + 68x^{\wedge}72 - 82x^{\wedge}70 + 84x^{\wedge}68 -$

$72x\hat{}66 + 56x\hat{}64 - 42x\hat{}62 - 5x\hat{}60 + 46x\hat{}58 - 91x\hat{}56 + 113x\hat{}54 - 139x\hat{}52 + 155x\hat{}50 - 138x\hat{}48 + 137x\hat{}46 - 100x\hat{}44 + 91x\hat{}42 - 46x\hat{}40 + 39x\hat{}38 - 30x\hat{}36 - x\hat{}32 - 28x\hat{}30 + 14x\hat{}28 - 60x\hat{}26 + 65x\hat{}24 - 93x\hat{}22 + 124x\hat{}20 - 136x\hat{}18 + 151x\hat{}16 - 124x\hat{}14 + 132x\hat{}12 - 96x\hat{}10 + 62x\hat{}8 - 13x\hat{}6 - 16x\hat{}4 + 42x\hat{}2 - 68 + 77x\hat{} - 2 - 77x\hat{} - 4 + 65x\hat{} - 6 - 54x\hat{} - 8 + 33x\hat{} - 10 - 20x\hat{} - 12 + 11x\hat{} - 14 - 5x\hat{} - 16 - x\hat{} - 18 + 3x\hat{} - 20 + x\hat{} - 24 - x\hat{} - 26$

4. Final remarks. The first author had developed a computer software for Apple Macintosh computers to assist researchers in Knot theory and recently have implemented in it an ability to calculate $X_L^{(3)}(q, q^3)^*$ for every 3 and 4 braid L. In particular, the software can distinguish the knots KT and KC in about 20 minutes on Macintosh Quadra 800 with 16 Mega bytes main memory.

Futhermore we have confirmed by direct matrix calculations that the matrixes obtained by our method satisfy all the defining relations of the braid groups B_3 and B_4.

The software has been arranged for the ftp network server, where the network address is wuarchive.wustl.edu and which are managed by professor Earl D. Fife of Calvin College whose *e*-mail address is fife@calvin. edu.

Our computer program is written in *C* language and is available as a complete description or else on disks with lists of all *W*-graphs corresponding to $H(q, n)$ for n up to 12 and all representation matrixes to compute $X_L^{(3)}(q, q^3)^*$ for every link L whose form has braid index of 3 and 4.

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