56. *The Central Limit Theorem for Rademacher System*

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1. Introduction. Let (Ω, P) be the Lebesgue probability space, i.e., $\Omega := [0,1)$ and P is Lebesgue measure. In this note, we regard any function on Ω as a function on $\boldsymbol{R}$ with period 1. The Rademacher system is a system $\{r_i\}$ of random variables on (Ω, P) defined by

$$r_1(\omega) := -1_{[0,1/2)}(\omega) + 1_{[1/2,1)}(\omega) \text{ and } r_i(\omega) := r_1(2^{i-1}\omega), \quad (i \geq 2).$$

Note that $(r_i + 1)/2$ gives the i-th digit of the dyadic expansion of a real number ω. Since $\{r_i\}$ is i.i.d., the De Moivre-Laplace theorem claims that the law of

$$X^{(m)}(\omega) := \frac{1}{\sqrt{m}} \sum_{i=1}^{m} r_i(\omega)$$

converges weakly to the standard normal distribution, as $m \to \infty$.

We put $X_n^{(m,\alpha)}(\omega) := X^{(m)}(\omega + n\alpha)$ for $n \in \boldsymbol{Z}$ and $\alpha \in \boldsymbol{R}$, and study the limit behaviour of the sequence $\{X_n^{(m,\alpha)}\}_{n\in\boldsymbol{Z}}$ as $m \to \infty$. Since this sequence is given by iterating the Weyl automorphism, it is stationary and, in most cases, dependent. Having studied the quasi-Monte Carlo method, Sugita [6] conjectured that the dependence disappears as $m \to \infty$, for almost all α. He proved that, for almost all α,

$$R^{(m,\alpha)}(k) := E(X_n^{(m,\alpha)} X_{n+k}^{(m,\alpha)}) = o(m^{-\beta}) \text{ as } m \to \infty, \ (k \in \boldsymbol{N}, \ 0 < \beta < 1/2).$$

We prove the following results related to the conjecture.

Theorem 1. *For almost all α with respect to Lebesgue measure, any finite dimensional distribution of $\{X_n^{(m,\alpha)}\}_{n\in\boldsymbol{Z}}$ converges weakly to the multi-dimensional standard normal law as $m \to \infty$; i.e., for all $n \in \boldsymbol{Z}$ and $k \in \boldsymbol{N}$,*

$$(X_n^{(m,\alpha)}, \ldots, X_{n+k-1}^{(m,\alpha)}) \xrightarrow{\mathcal{D}} N(\mathbf{0}, I_k) \text{ as } m \to \infty. \tag{1.1}$$

Here $\xrightarrow{\mathcal{D}}$ denotes convergence in law, and I_k the k-dimensional unit matrix.

Theorem 2. *For any α, the correlation $R^{(m,\alpha)}(k) := E(X_n^{(m,\alpha)} X_{n+k}^{(m,\alpha)})$ is given by*

$$R^{(m,\alpha)}(k) = \frac{1}{m} \sum_{i=1}^{m} \varphi(2^{i-1}k\alpha), \text{ where } \varphi(x) := |4x - 2| - 1, \ (x \in \Omega). \tag{1.2}$$

Moreover, for any $k \in \boldsymbol{N}$ and for almost all $\alpha \in \boldsymbol{R}$ with respect to Lebesgue measure, it holds that

$$\limsup_{m\to\infty} \sqrt{\frac{m}{\log\log m}}\, R^{(m,\alpha)}(k) = \sqrt{\frac{2}{3}}. \tag{1.3}$$

Theorem 3. *The Hausdorff dimension of the set of α for which* (1.1) *does not hold is* 1.

Remark. We can improve Theorem 3 as follows: The Hausdorff dimension of the set of α such that finite dimensional distribution of $\{X_n^{(m,\alpha)}\}_{n\in\boldsymbol{Z}}$ converges to that of some stationary dependent gaussian sequence is 1. The

proof will be given in a forthcoming paper.

2. Proof of Theorem 1. First we prove two lemmas.

Lemma 1. *For any sequence $\{\alpha_i\}$ of real numbers, the sequence $\{r_i'(\omega) := r_i(\omega + \alpha_i)\}$ is an i.i.d. on (Ω, P).*

Proof. Clearly, we have $P(r_i' = -1) = P(r_i' = 1) = 1/2$. To prove the independence, it is sufficient to prove

$$(2.1) \qquad P(r_1' = \varepsilon_1, \dots, r_n' = \varepsilon_n) \leq \frac{1}{2^n} \quad (\varepsilon_i = \pm 1,\ i = 1, \dots, n,\ n \in \boldsymbol{N}).$$

Actually, from (2.1), we have

$$1 = \sum_{\varepsilon_i = \pm 1, (i=1,\dots,n)} P(r_1' = \varepsilon_1, \dots, r_n' = \varepsilon_n) \leq \sum \frac{1}{2^n} = 1,$$

and thereby we see that the equality in (2.1) holds.

Take $\varepsilon_i = \pm 1$ arbitrarily and put $A_i := \{\omega \in \Omega : r_i'(\omega) = \varepsilon_i\}$. The next property of A_i is easily verified: If $\omega, \omega' \in \Omega$ satisfy $\omega \equiv \omega' + (2j + 1)/2^i \pmod 1$ for some $j \in \boldsymbol{Z}$, then either $\omega \in A_i$ or $\omega' \in A_i$.

Now, we define a mapping $T_n : \Omega \to [0, 1/2^n)$ by $T_n(\omega) := (2^n\omega - \lfloor 2^n\omega \rfloor)/2^n$. If A is a measurable subset of Ω such that the restriction $T_n|_A$ of T_n to A is injective, then $T_n|_A$ is obviously measure-preserving and thereby the inequality $P(A) = P(T_n(A)) \leq 1/2^n$ holds.

Here, we prove that $T_n|_{A_1 \cap \dots \cap A_n}$ is injective, from which (2.1) follows. The proof is by contradiction. Suppose that $\omega, \omega' \in A_1 \cap \dots \cap A_n$ satisfy $\omega < \omega'$ and $T_n(\omega) = T_n(\omega')$. Then there exists a $k \in [1, 2^n) \cap \boldsymbol{Z}$ such that $\omega' = \omega + k/2^n$. Factoring k, we have $k = 2^h(2j + 1)$ ($h \in [0, n) \cap \boldsymbol{Z}$ and $j \in \boldsymbol{Z}$), from which $\omega' = \omega + (2j + 1)/2^{n-h}$ follows. As we mentioned before, either $\omega \in A_{n-h}$ or $\omega' \in A_{n-h}$. This contradicts the assumption $\omega, \omega' \in A_1 \cap \dots \cap A_n$.

Lemma 2. *Suppose that $p, q \in \boldsymbol{Z}$ satisfy $p < q$. Then, for almost all α with respect to Lebesgue measure,*

$$(2.2) \qquad \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} r_i(\omega + p\alpha) r_i(\omega + q\alpha) = 0, \quad P\text{-}a.e.\ \omega.$$

Proof. Since the sequence $\{r_i(\omega_1) r_i(\omega_2)\}$ is an i.i.d. on $(\Omega \times \Omega, P \times P)$, by the law of large numbers, we have

$$\lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} r_i(\omega_1) r_i(\omega_2) = 0, \quad P \times P\text{-}a.e.\ (\omega_1, \omega_2).$$

Since r_i is periodic, the Lebesgue measure of $\boldsymbol{R}^2 \setminus M$ is 0, where M is given by

$$M := \left\{ (\omega_1, \omega_2) \in \boldsymbol{R}^2 \,\middle|\, \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} r_i(\omega_1) r_i(\omega_2) = 0 \right\}.$$

Since the linear transformation $T : (\omega, \alpha) \mapsto (\omega + p\alpha, \omega + q\alpha)$ is regular, the Lebesgue measure of $T^{-1}(\boldsymbol{R}^2 \setminus M)$ is 0. Therefore, for almost all (ω, α),

$$\lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} r_i(\omega + p\alpha) r_i(\omega + q\alpha) = 0.$$

By Fubini's theorem, we have the conclusion.

To prove Theorem 1, we use the next theorem due to McLeish [5].

Theorem A. *Let $\{\zeta_{m,j}\,;\, 1 \leq j \leq k_m\}$ be a triangular array of random vari-*

ables and put $L_m := \Pi_{j \le k_m}(1 + \sqrt{-1}\, t\zeta_{m,j})$. *The law of* $\sum_{j \le k_m} \zeta_{m,j}$ *converges to the standard normal distribution as* $m \to \infty$, *provided that the following four conditions are satisfied for all* $t \in \boldsymbol{R}$:

(1) $EL_m \to 1$ *as* $m \to \infty$;

(2) *The sequence* $\{L_m\}_{m \in \mathbf{N}}$ *is uniformly integrable*;

(3) $\sum_{j \le k_m} \zeta_{m,j}^2 \to 1$ *in probability as* $m \to \infty$;

(4) $\max_{j \le k_m} |\zeta_{m,j}| \to 0$ *in probability as* $m \to \infty$.

Because of Lemma 2, it is sufficient for us to prove (1.1) assuming (2.2) for all $p < q$. We put $n = 1$ in (1.1), since the sequence is stationary. We prove that, for any $a_1, \ldots, a_k$ satisfying $a_1^2 + \cdots + a_k^2 = 1$, the law of $a_1 X_1^{(m,\alpha)} + \cdots + a_k X_k^{(m,\alpha)}$ converges to the standard normal law. Because of Cramér-Wold's theorem (Theorem 7.7 in Billingsley [2]), this is equivalent to (1.1).

Putting $\eta_i(\omega) := \sum_{j=1}^k a_j r_i(\omega + j\alpha)$, we have

$$a_1 X_1^{(m,\alpha)} + \cdots + a_k X_k^{(m,\alpha)} = \frac{1}{\sqrt{m}} \sum_{i=1}^m \eta_i.$$

We note the following two properties of $\{\eta_i\}$:

(a) $|\eta_i| \le k$;

(b) $E(\eta_{i_1} \ldots \eta_{i_s}) = 0$, for any $i_1 < \cdots < i_s$ and $s \in \boldsymbol{N}$.

(a) is trivial. (b) is verified by expanding $\eta_{i_1} \ldots \eta_{i_s}$ into a linear combination of

$$r_{i_1}(\omega + j_1\alpha) \ldots r_{i_s}(\omega + j_s\alpha), \quad (j_t = 1, \ldots, k,\ t = 1, \ldots, s),$$

and by noting that the expectations of these are 0 because of Lemma 1.

Putting $k_m := m$ and $\zeta_{m,j} := \eta_j/\sqrt{m}$, we apply Theorem A to prove the convergence. The four conditions are easily verified: $EL_m = 1$ follows from (b); By (a), we have $|L_m| \le (1 + t^2k^2/m)^{m/2} \le e^{t^2k^2/2}$, which implies (2); Since $\sum_{j \le k_m} \zeta_{m,j}^2(\omega) - 1$ is expanded into a linear combination of the sums in (2.2), (3) follows from the assumption; (4) is clear from (a).

3. Proof of Theorem 2. By using Lemma 1, we have

$$R^{(m,\alpha)}(k) = EX_0^{(m,\alpha)} X_k^{(m,\alpha)} = \frac{1}{m} \sum_{i=1}^m E(r_i(\omega) r_i(\omega + k\alpha)).$$

An easy calculation gives $E(r_1(\omega) r_1(\omega + k\alpha)) = \varphi(k\alpha)$. Since the dyadic transformation is measure preserving, we have

$$\begin{aligned} E(r_i(\omega) r_i(\omega + k\alpha)) &= E(r_1(2^{i-1}\omega) r_1(2^{i-1}\omega + 2^{i-1}k\alpha)) \\ &= E(r_1(\omega) r_1(\omega + 2^{i-1}k\alpha)) \\ &= \varphi(2^{i-1}k\alpha). \end{aligned}$$

These prove (1.2). To prove (1.3), we apply the following law of the iterated logarithm due to Maruyama [4]:

$$\limsup_{m \to \infty} \frac{1}{\sqrt{2m \log\log m}} \sum_{i=1}^m \varphi(2^{i-1}x) = \sigma, \quad \text{a.e. } x,$$

where

$$\sigma^2 = \int_0^1 \varphi^2(x)\,dx + 2 \sum_{j=1}^\infty \int_0^1 \varphi(x)\varphi(2^j x)\,dx = \frac{1}{3}.$$

The last evaluation of σ^2 is easily given by $\varphi(x) = 2r_1(x) \sum_{j=2}^\infty 2^{-j} r_j(x)$.

4. Proof of Theorem 3. We follow the method of Hawkes (Proof of

Theorem 4 of [3]). We introduce a Bernoulli i.i.d. $\{Y_n\}$ on another probability space (Ω_0, P_0) such that $P_0\{Y_n = 0\} = q$ and $P_0\{Y_n = 1\} = p$, $(p + q = 1)$. Let μ_p be the law of $\sum_{j=1}^{\infty} 2^{-j} Y_j$. We can easily verify that μ_p is a probability measure on Ω and that, on the probability space (Ω, μ_p), the dyadic transformation $\omega \mapsto 2\omega$ is measure-preserving and ergodic.

Because of (1.2) and the ergodicity, we have

$$R^{(m,\alpha)}(1) \to \int_{\Omega} \varphi(x)\mu_p(dx), \text{ as } m \to \infty, \ \mu_p\text{-a.e. } \alpha.$$

The last integral is evaluated as follows: By the definition of μ_p,

$$\begin{aligned}\int_{\Omega} \varphi(x)\mu_p(dx) &= E_{P_0}\varphi\Big(\sum_{j=1}^{\infty} 2^{-j} Y_j\Big) \\ &= \sum_{i=0}^{1} E_{P_0}\varphi\Big(\frac{i}{2} + \sum_{j=2}^{\infty} 2^{-j} Y_j\Big) P_0\{Y_1 = i\}\ ;\end{aligned}$$

Since $0 \leq \sum_{j=2}^{\infty} 2^{-j} Y_j \leq 1/2$, we have $\int_{\Omega} \varphi(x)\mu_p(dx) = (1 - 2p)^2$.

Now we recall the following result by Billingsley ([1] p. 141 onwards): Hausdorff dimension of $A \subset \Omega$ is equal to or greater than $e_p := -(p \log_2 p + q \log_2 q)$ if $\mu_p(A) > 0$. Then, we see that $R^{(m,\alpha)}(1) \to (1 - 2p)^2$ holds on the set of α whose dimension is at least e_p.

$E(X_n^{(m,\alpha)})^4 \leq 4$ implies the uniform integrability of $\{X_0^{(m,\alpha)} X_1^{(m,\alpha)}\}_{m \in \mathbf{N}}$. Thus, $R^{(m,\alpha)}(1) \to 0$ follows from (1.1).

Therefore, the dimension of the set of α such that (1.1) does not hold is at least $e_p (p \neq 1/2)$. Taking the supremum, we have the conclusion.

References

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