55. *Complements to the Furuta Inequality*[†)]

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Abstract: Complementary results to the Furuta inequality are given in cases of positive invertible operators.

§1. Introduction. In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (in symbol: $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. Also an operator T is strictly positive (in symbol: $T > 0$) if T is positive and invertible.

As an extension of the Löwner-Heinz theorem [12][10], we established the following Furuta inequality [4].

Theorem A (Furuta inequality). *If $A \geq B \geq 0$, then for each $r \geq 0$,*

(i) $$(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q}$$

and

(ii) $$(A^r A^p A^r)^{1/q} \geq (A^r B^p A^r)^{1/q}$$

hold for p and q such that $p \geq 0$ and $q \geq 1$ with $(1+2r)q \geq p+2r$.

Alternative proofs of Theorem A are given in [1][5] and [11] and also one page proof is shown in [6]. Recently it turns out that Theorem A has a lot of applications, in fact [2][3][7][8] and [9] are some of them.

We remark that the Furuta inequality yields the following famous Löwner-Heinz inequality when we put $r=0$ in (i) or (ii) of Theorem A;

Theorem B (Löwner-Heinz inequality).

(*) $$A \geq B \geq 0 \text{ ensures } A^\alpha \geq B^\alpha \text{ for any } \alpha \in [0,1].$$

§2. Statement of results. Theorem 1. *If $A \geq B > 0$, then*

$$(B^\gamma A^\alpha B^\gamma)^\beta \geq (B^\gamma B^\alpha B^\gamma)^\beta$$

holds under any one of the following conditions;

(i) $$\frac{1}{\beta} \leq \alpha,\ 0 < \beta < 1, \text{ and } \gamma = \frac{\alpha\beta - 1}{2(1-\beta)}$$

(ii) $$\frac{1}{\beta} \leq \alpha \leq 1,\ 1 < \beta \leq 2, \text{ and } \gamma = \frac{\alpha\beta - 1}{2(1-\beta)}$$

(iii) $$\frac{1}{2} \leq \alpha \leq 1,\ 2 \leq \beta, \text{ and } \gamma = \frac{\alpha\beta - 1}{2(1-\beta)}.$$

Remark 1. (i) and (ii) are announced in [13, p. 61], but in the proof of Theorem 1 under below we remark that (i) is nothing but exchange of parameters p, q and r in Theorem A and a simple proof of (ii) can be obtained along a method of [6] by using polar decomposition. In this paper we shall

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show (iii). We have to assume invertibility of A and B in the cases (ii) and (iii) since $\gamma \le 0$.

We cite the following known result to give a proof of Theorem 1.

Lemma A [7]. *Let A and B be positive invertible operators. For any real number r,*

$$(BAB)^r = BA^{1/2}(A^{1/2}B^2A^{1/2})^{r-1}A^{1/2}B.$$

Lemma 1. *Let* $\frac{1}{2} \le \alpha \le 1$, $2n \le \beta \le 2n+1$ *for some natural number* n *and* $\gamma = \frac{\alpha\beta - 1}{2(1-\beta)}$. *Then the following* (1) *and* (2) *hold;*

(1) $x_j = (\alpha + 2\gamma)(\beta - 2n) + 2\gamma + (2\alpha + 4\gamma)j \in [-1,0]$ *for* $j = 0,1,2,\dots,n-1$.

(2) $y_j = (\alpha + 2\gamma)(\beta - 2n) + 2\gamma + (2\alpha + 4\gamma)j + 2\alpha \in [0,1]$ *for* $j = 0,1,2,\dots,n-1$.

Lemma 2. *Let* $\frac{1}{2} \le \alpha \le 1$, $2n+1 \le \beta \le 2(n+1)$ *for some natural number* n *and* $\gamma = \frac{\alpha\beta - 1}{2(1-\beta)}$. *Then the following* (3) *and* (4) *hold;*

(3) $x_j = (\alpha + 2\gamma)(\beta - 2n - 1) + \alpha + (2\alpha + 4\gamma)j \in [0,1]$ *for* $j = 0,1,2,\dots,n-1,n$.

(4) $y_j = (\alpha + 2\gamma)(\beta - 2n - 1) + \alpha + (2\alpha + 4\gamma)j + 4\gamma \in [-1,0]$ *for* $j = 0,1,2,\dots,n-1$.

Proof of Lemma 1. (1) It turns out that $x_0 \le x_1 \le \cdots \le x_{n-1}$ by the definition of x_j and $\alpha + 2\gamma \ge 0$. On the other hand $x_{n-1} = 1 - 2\alpha \le 0$ since $\frac{1}{2} \le \alpha$ and $x_0 = (\alpha + 2\gamma)(\beta - 2n) + 2\gamma + 1 - 1 = \frac{2(1-\alpha)(\beta - n)}{\beta - 1} - 1 \ge -1$. Hence $x_j \in [-1,0]$ for $j = 0,1,2,\dots,n-1$.

(2) $y_0 \le y_1 \le \cdots \le y_{n-2} \le y_{n-1}$ since $y_j = x_j + 2\alpha$ and $x_0 \le x_1 \le \cdots \le x_{n-2} \le x_{n-1}$ stated in the proof of (1). $y_j = x_j + 2\alpha \ge -1 + 2\alpha \ge 0$ since $x_j \ge -1$ by (1) and $y_{n-1} = 1$ since $y_{n-1} = x_{n-1} + 2\alpha = 1 - 2\alpha + 2\alpha = 1$. Hence $y_j \in [0,1]$ for $j = 0,1,2,\dots,n-1$.

Proof of Lemma 2. (3) $x_0 \le x_1 \le x_2 \le \cdots \le x_n$ by the definition of x_j and $\alpha + 2\gamma \ge 0$. On the other hand $x_0 = (\alpha + 2\gamma)(\beta - 2n - 1) + \alpha \ge \alpha \ge \frac{1}{2}$ and $x_n = 1$. Hence $x_j \in [\alpha, 1] \subseteq [0,1]$ for $j = 0,1,2,\dots,n-1,n$.

(4) $y_0 \le y_1 \le \cdots \le y_{n-2} \le y_{n-1}$ since $y_j = x_j + 4\gamma$ and $x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n$ stated in the proof of (3). $y_0 = (\alpha + 2\gamma)(\beta - 2n - 1) + \alpha + 4\gamma \ge \alpha + 4\gamma \ge -1$ since $\alpha + 4\gamma + 1 = \frac{(1-\alpha)(1+\beta)}{\beta - 1} \ge 0$ and $y_{n-1} = 1 - 2\alpha \le 0$. Hence $y_{j-1} \in [-1,0]$ for $j = 0,1,2,\dots,n-1$.

Proof of Theorem 1. (i) Put $(1+2r)q = p + 2r$ for $r \ge 0$, $p \ge 1$ and $q \ge 1$ in Theorem A, then we easily obtain $p \ge q \ge 1$ and we have only to replace p by α, r by γ and $1/q$ by β.

(ii) First of all, we easily obtain the following (5), (6) and (7):

$$2\gamma \in [-1,0], \tag{5}$$

(6) $$\beta - 1 \in [0,1],$$
(7) $$\alpha + (\alpha + 2\gamma)(\beta - 1) = 1.$$

Then we have

$$\begin{aligned}(B^\gamma A^\alpha B^\gamma)^\beta &= B^\gamma A^{\alpha/2}(A^{\alpha/2}B^{2\gamma}A^{\alpha/2})^{\beta-1}A^{\alpha/2}B^\gamma \text{ by Lemma A}\\ &\geq B^\gamma A^{\alpha/2}(A^{\alpha/2}A^{2\gamma}A^{\alpha/2})^{\beta-1}A^{\alpha/2}B^\gamma \text{ by (5), (6) and (*)}\\ &= B^\gamma A^{\alpha+(\alpha+2\gamma)(\beta-1)}B^\gamma\\ &= B^\gamma A^1 B^\gamma \text{ by (7)}\\ &\geq B^{1+2\gamma} = B^{(\alpha+2\gamma)\beta} \text{ by (7).}\end{aligned}$$

(iii) (a) In the first case $2n \leq \beta \leq 2n+1$ for some natural number n.

$$\begin{aligned}&(B^\gamma A^\alpha B^\gamma)^\beta = (B^\gamma A^\alpha B^\gamma)^n(B^\gamma A^\alpha B^\gamma)^{\beta-2n}(B^\gamma A^\alpha B^\gamma)^n\\ &\geq (B^\gamma A^\alpha B^\gamma)^n(B^\gamma B^\alpha B^\gamma)^{\beta-2n}(B^\gamma A^\alpha B^\gamma)^n \text{ by } \alpha \in \left[\frac{1}{2}, 1\right] \text{ and (*)}\\ &= (B^\gamma A^\alpha B^\gamma)^{n-1}B^\gamma A^\alpha B^\gamma B^{(\alpha+2\gamma)(\beta-2n)}B^\gamma A^\alpha B^\gamma(B^\gamma A^\alpha B^\gamma)^{n-1}\\ &= (B^\gamma A^\alpha B^\gamma)^{n-1}B^\gamma A^\alpha B^{2\gamma+(\alpha+2\gamma)(\beta-2n)}A^\alpha B^\gamma(B^\gamma A^\alpha B^\gamma)^{n-1}\\ &\geq (B^\gamma A^\alpha B^\gamma)^{n-1}B^\gamma A^\alpha A^{2\gamma+(\alpha+2\gamma)(\beta-2n)}A^\alpha B^\gamma(B^\gamma A^\alpha B^\gamma)^{n-1} \text{ by (1)}\\ &= (B^\gamma A^\alpha B^\gamma)^{n-1}B^\gamma A^{2\alpha+2\gamma+(\alpha+2\gamma)(\beta-2n)}B^\gamma(B^\gamma A^\alpha B^\gamma)^{n-1}\\ &\geq (B^\gamma A^\alpha B^\gamma)^{n-1}B^\gamma B^{2\alpha+2\gamma+(\alpha+2\gamma)(\beta-2n)}B^\gamma(B^\gamma A^\alpha B^\gamma)^{n-1} \text{ by (2)}\\ &= (B^\gamma A^\alpha B^\gamma)^{n-2}B^\gamma A^\alpha B^{(4\gamma+2\alpha)1+2\gamma+(\alpha+2\gamma)(\beta-2n)}A^\alpha B^\gamma(B^\gamma A^\alpha B^\gamma)^{n-2}\\ &= (B^\gamma A^\alpha B^\gamma)^{n-2}B^\gamma A^\alpha A^{(4\gamma+2\alpha)1+2\gamma+(\alpha+2\gamma)(\beta-2n)}A^\alpha B^\gamma(B^\gamma A^\alpha B^\gamma)^{n-2} \text{ by (1)}\\ &\cdots\cdots\\ &\geq (B^\gamma A^\alpha B^\gamma)^{n-j}B^\gamma A^{(4\gamma+2\alpha)(j-1)+2\gamma+2\alpha+(\alpha+2\gamma)(\beta-2n)}B^\gamma(B^\gamma A^\alpha B^\gamma)^{n-j} \text{ by (1) and (2)}\\ &\cdots\cdots\\ &\geq (B^\gamma A^\alpha B^\gamma)^{n-k}B^\gamma A^\alpha B^{(4\gamma+2\alpha)(k-1)+2\gamma+(\alpha+2\gamma)(\beta-2n)}A^\alpha B^\gamma(B^\gamma A^\alpha B^\gamma)^{n-k} \text{ by (1) and (2)}\\ &\cdots\cdots\\ &\geq B^\gamma A^{(4\gamma+2\alpha)(n-1)+2\alpha+2\gamma+(\alpha+2\gamma)(\beta-2n)}B^\gamma \text{ by (1) and (2)}\\ &= B^\gamma A^1 B^\gamma \geq B^{1+2\gamma} = B^{(\alpha+2\gamma)\beta} \text{ by } y_{n-1} = 1 \text{ in (2).}\end{aligned}$$

Thus the proof of the first case (a) is complete.

(b) In the second case $2n+1 \leq \beta \leq 2(n+1)$ for some natural number n.

$$\begin{aligned}&(B^\gamma A^\alpha B^\gamma)^\beta = (B^\gamma A^\alpha B^\gamma)^n(B^\gamma A^\alpha B^\gamma)^{\beta-2n}(B^\gamma A^\alpha B^\gamma)^n\\ &= (B^\gamma A^\alpha B^\gamma)^n B^\gamma A^{\alpha/2}(A^{\alpha/2}B^{2\gamma}A^{\alpha/2})^{\beta-2n-1}A^{\alpha/2}B^\gamma(B^\gamma A^\alpha B^\gamma)^n \text{ by Lemma A}\\ &\geq (B^\gamma A^\alpha B^\gamma)^n B^\gamma A^{\alpha/2}(A^{\alpha/2}A^{2\gamma}A^{\alpha/2})^{\beta-2n-1}A^{\alpha/2}B^\gamma(B^\gamma A^\alpha B^\gamma)^n \text{ by } 2\gamma \in [-1,0] \text{ and (*)}\\ &= (B^\gamma A^\alpha B^\gamma)^n B^\gamma A^{\alpha+(\alpha+2\gamma)(\beta-2n-1)}B^\gamma(B^\gamma A^\alpha B^\gamma)^n\\ &\geq (B^\gamma A^\alpha B^\gamma)^n B^\gamma B^{\alpha+(\alpha+2\gamma)(\beta-2n-1)}B^\gamma(B^\gamma A^\alpha B^\gamma)^n \text{ by (3)}\\ &= (B^\gamma A^\alpha B^\gamma)^{n-1}B^\gamma A^\alpha B^{4\gamma+\alpha+(\alpha+2\gamma)(\beta-2n-1)}A^\alpha B^\gamma(B^\gamma A^\alpha B^\gamma)^{n-1}\\ &\geqq (B^\gamma A^\alpha B^\gamma)^{n-1}B^\gamma A^\alpha A^{4\gamma+\alpha+(\alpha+2\gamma)(\beta-2n-1)}A^\alpha B^\gamma(B^\gamma A^\alpha B^\gamma)^{n-1} \text{ by (4)}\\ &= (B^\gamma A^\alpha B^\gamma)^{n-1}B^\gamma A^{2\alpha+4\gamma+\alpha+(\alpha+2\gamma)(\beta-2n-1)}B^\gamma(B^\gamma A^\alpha B^\gamma)^{n-1}\\ &\geq (B^\gamma A^\alpha B^\gamma)^{n-1}B^\gamma B^{2\alpha+4\gamma+\alpha+(\alpha+2\gamma)(\beta-2n-1)}B^\gamma(B^\gamma A^\alpha B^\gamma)^{n-1} \text{ by (3)}\\ &= (B^\gamma A^\alpha B^\gamma)^{n-2}B^\gamma A^\alpha B^{4\gamma+(2\alpha+4\gamma)1+\alpha+(\alpha+2\gamma)(\beta-2n-1)}A^\alpha B^\gamma(B^\gamma A^\alpha B^\gamma)^{n-2}\\ &\geq (B^\gamma A^\alpha B^\gamma)^{n-2}B^\gamma A^\alpha A^{4\gamma+(2\alpha+4\gamma)1+\alpha+(\alpha+2\gamma)(\beta-2n-1)}A^\alpha B^\gamma(B^\gamma A^\alpha B^\gamma)^{n-2} \text{ by (4)}\\ &= (B^\gamma A^\alpha B^\gamma)^{n-2}B^\gamma A^{(2\alpha+4\gamma)2+\alpha+(\alpha+2\gamma)(\beta-2n-1)}B^\gamma(B^\gamma A^\alpha B^\gamma)^{n-2}\\ &\cdots\cdots\\ &\geq (B^\gamma A^\alpha B^\gamma)^{n-j}B^\gamma A^{(2\alpha+4\gamma)j+\alpha+(\alpha+2\gamma)(\beta-2n-1)}B^\gamma(B^\gamma A^\alpha B^\gamma)^{n-j} \text{ by (3) and (4)}\\ &\cdots\cdots\end{aligned}$$

$= (B^r A^\alpha B^r)^{n-k} B^r A^\alpha A^{(2\alpha+4r)(k-1)+4r+\alpha+(\alpha+2r)(\beta-2n-1)} A^\alpha B^r (B^r A^\alpha B^r)^{n-k}$

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$\geq B^r A^{(2\alpha+4r)n+\alpha+(\alpha+2r)(\beta-2n-1)} B^r$ by (3) and (4)
$= B^r A^1 B^r = B^{(\alpha+2r)\beta}$ by $x_n = 1$ of (3).
Thus the proof of case (b) is complete.

Finally the proof of (iii) in Theorem 1 is complete together with case (a) and case (b).

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