49. *On a Diophantine Equation of Erdös*

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Introduction. Let n, k, l, x be integers greater than 1. Erdös [4] showed that if $k > 3$, then the Diophantine equation

$$\binom{n}{k} = x^l \qquad (n \geq 2k) \tag{1}$$

has no solutions, and conjectured that if $l \geq 3$, then the equation (1) has no solutions (cf. Guy [6], section D17, p. 97). The condition $n \geq 2k$ involves no loss of generality since $\binom{n}{k} = \binom{n}{n-k}$.

On the other hand, it is well known that the equation $\binom{n}{2} = x^2$ has infinitely many solutions and that the only solution of the equation $\binom{n}{3} = x^2$ is $n = 50$, $x = 140$, as shown by Watson [14] and Ljunggren [8] independently (cf. Guy [6], section D3, p. 82). In 1974, Tijdeman proved, by means of Baker's method, that the remaining equations

$$\binom{n}{2} = x^l \qquad (l \geq 3) \tag{2}$$

and

$$\binom{n}{3} = x^l \qquad (l \geq 3) \tag{3}$$

have only finitely many solutions respectively and these can be effectively determined, which follows immediately from the following:

Theorem (Tijdeman [13]). *Let $P(x)$ be a polynomial with rational coefficients and with at least two simple rational zeros. If $m = 2$ we further assume that P has a third simple zero. Then the equation*

$$y^m = P(x)$$

has only finitely many solutions in integers $m \geq 2$, $x \geq 1$, $y \geq 2$ and these can be effectively determined.

But Tijdeman gave explicitly no upper bounds for solutions of (2) and (3).

The purpose of this paper is to give explicit upper bounds for these solutions, by use of the recent results concerning Baker theory. Indeed, we prove the following:

Theorem 1. *All solutions of the equation* (2) *satisfy*

$$l < 4250 \quad \textit{and} \quad x < e^{10^{172}}.$$

Theorem 2. *All solutions of the equation* (3) *satisfy*

$$l < 4250 \quad \textit{and} \quad x < e^{3\cdot 10^{172}}.$$

In §3, we shall show that some other methods than Baker's can also be used for treating the equation (2).

§1. Proof of Theorem 1. Suppose that the equation (2) has solutions. Then we have $\frac{n(n-1)}{2} = x^l$, so

$$\begin{cases} n & = x_1^l \\ n-1 & = 2x_2^l \end{cases} \quad \text{or} \quad \begin{cases} n & = 2x_1^l \\ n-1 & = x_2^l \end{cases}$$

where x_1, x_2 are positive integers with $x = x_1 x_2$. Subtracting both sides respectively yields

$$X^l - 2Y^l = \pm 1, \tag{4}$$

where $X = x_1$, $Y = x_2$ or $X = x_2$, $Y = x_1$. We note that since $x = x_1 x_2$, we have

$$x < X^2.$$

Using now Baker theory, we give upper bounds for solutions of (4). We first prepare two lemmas.

Let α be an algebraic number of degree d with the minimal polynomial

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (x - \alpha_i),$$

where the a_i's are relatively prime integers with $a_0 > 0$ and the α_i's are conjugates of α. Then

$$h(\alpha) = \frac{1}{d} (\log a_0 + \sum_{i=1}^{d} \log \max(1, |\alpha_i|))$$

is called the *absolute logarithmic height* of α. In particular, if $\alpha \in \boldsymbol{Q}$, say $\alpha = p/q$ as a fraction in lowest terms, then we have

$$h\left(\frac{p}{q}\right) = \log \max(|p|, |q|).$$

Lemma 1 (Maohua [9]). *Let α_1, α_2 be real algebraic numbers with $\alpha_1 \geq 1$, $\alpha_2 \geq 1$, and let D denote the degree of $\boldsymbol{Q}(\alpha_1, \alpha_2)$. Let b_1, b_2 be positive integers, and let $b = b_1/Dh(\alpha_2) + b_2/Dh(\alpha_1)$. Further, let T be any number satisfying $1 \leq T \leq 0.52 + \log b$. If $\Lambda = b_1 \log \alpha_1 - b_2 \log \alpha_2 \neq 0$, then we have*

$$|\Lambda| \geq \exp\left\{-70\left(1 + \frac{0.1137}{T}\right)^2 D^4 h(\alpha_1) h(\alpha_2) (0.52 + \log b)^2\right\}.$$

Lemma 2 (Győry and Papp [7]). *Let $f(x, y)$ be an irreducible binary form with integer coefficients and degree $d \geq 3$. Let m be a non-zero integer. All integral solutions x, y of the Thue equation*

$$f(x, y) = m$$

satisfy

$$\log \max(|x|, |y|) \leq 10^{150} d^6 \log(HM),$$

where $M = |m|$ and H is the maximum absolute value of coefficients of $f(x, y)$.

We use Lemmas 1,2 to prove the following:

Proposition 1. *All solutions of the equation* (4) *satisfy*

$$l < 4250 \quad \text{and} \quad Y \leq X < e^{5 \cdot 10^{171}}.$$

Proof. The equation (4) clearly has a solution $X = 1$, $Y = 1$ and no solution $X = 2$. Thus we may assume that $X > 2$.

From (4), we have

$$\frac{2Y^l}{X^l} = 1 \pm \frac{1}{X^l}.$$

Recalling that if $|\alpha| \leq \frac{1}{2}$ then $|\log(1+\alpha)| \leq 2|\alpha|$, we obtain

$$\left| l \log\left(\frac{X}{Y}\right) - \log 2 \right| \leq \frac{2}{X^l}. \tag{5}$$

On the other hand, since we may choose $T = 9$ if $l \geq 4000$, it follows from Lemma 1 that

$$\left| l \log\left(\frac{X}{Y}\right) - \log 2 \right| \geq \exp\{-71.8(\log X)(\log 2)(0.52 + \log b)^2\}, \tag{6}$$

where $b = l/\log 2 + 1/\log X$. Combining (5) with (6), we have

$$l \log X - \log 2 \leq 71.8(\log X)(\log 2)\left\{0.52 + \log\left(\frac{l}{\log 2} + \frac{1}{\log X}\right)\right\}^2,$$

so

$$l < 71.8(\log 2)\left\{0.52 + \log\left(\frac{l}{\log 2} + 1\right)\right\}^2 + 1.$$

Hence we obtain $l < 4250$.

It follows from Lemma 2 that

$$\begin{aligned} \log \max(X, Y) &\leq 10^{150} l^6 \log 2 \\ &< 5 \cdot 10^{171}. \end{aligned}$$

Therefore we have $Y \leq X < e^{5 \cdot 10^{171}}$. This proves Proposition 1.

Since $x < X^2$, Theorem 1 follows easily from Proposition 1.

§2. Proof of Theorem 2. Erdős [4] proved that if the equation (1) has solutions, then we have

$$n - i = c_i x_i^l$$

for all i with $0 \leq i < k$, where the c_i's are $1, 2, \ldots, k$ in some order and the x_i's are positive integers with $x = x_0 x_1 \cdots x_{k-1}$.

Hence, if the equation (3) has solutions, then there are positive integers x_r, x_s, x_t with $x = x_r x_s x_t$ such that

$$n - r = x_r^l, \quad n - s = 2x_s^l, \quad n - t = 3x_t^l,$$

where r, s, t are distinct integers with $0 \leq r, s, t < 3$. Thus we have $x_r^l - 2x_s^l = s - r = \pm 1$ or ± 2. Therefore we obtain

$$X^l - 2Y^l = \pm 1 \tag{4}$$

or

$$X^l - 2^{l-1} Y^l = \pm 1, \tag{7}$$

where $X = x_r$, $Y = x_s$ or $X = x_s$, $Y = x_r/2$. We note that since $x = x_r x_s x_t < x_r^3$, we have

$$x < X^3 \text{ or } x < (2Y)^3$$

according as (4) or (7).

As we gave upper bounds for solutions of (4) in Proposition 1, the following proposition gives those of (7).

Proposition 2. *All solutions of the equation* (7) *satisfy*

$$l < 4250 \quad \textit{and} \quad X < 2Y < e^{10^{172}}.$$

Proof. From (7), we have

$$\frac{2X^l}{(2Y)^l} = 1 \pm \frac{2}{(2Y)^l}.$$

By the same way as in the proof of Proposition 1, we obtain $l < 4250$.

Since the equation (7) becomes $(2Y)^l - 2X^l = \pm 2$, it follows from Lemma 2 that

$$\begin{aligned}\log \max (X, 2Y) &\leq 10^{150} l^6 \log 4 \\ &< 10^{172}.\end{aligned}$$

Therefore we have $X < 2Y < e^{10^{172}}$. This proves Proposition 2.

Since $x < X^3$ or $x < (2Y)^3$ according as (4) or (7), Theorem 2 follows easily from Propositions 1,2.

§3. Supplementary remarks. We used above Baker theory, i. e. that of linear forms in the logarithms of algebraic numbers to obtain upper bounds for solutions of the equation

$$X^l - 2Y^l = \pm 1. \tag{4}$$

We remark in this section that the following methods can also be used for treating (4).

(i) Quadratic reciprocity laws. We first note that the equation (4) can be written as

$$x^l + y^l = z^2,$$

where $x = \pm X$, $y = Y^2$, $z = \pm Y^l + 1$.

Rotkiewicz [10] showed the following theorem, applying Jacobi's symbol to Lehmer's numbers and using quadratic reciprocity laws.

Theorem A. *Let $(x, y) = 1$ and l be a prime > 3. If $l \mid z$, $2 \nmid z$ or $l \nmid z$, $2 \mid z$, then the equation $x^l + y^l = z^2$ has no solutions.*

This implies the theorem of Terjanian [12] which states that the first case of Fermat's Last Theorem for even exponents is true. We easily see that if $l \mid (Y \pm 1)$, $2 \mid Y$ or $l \nmid (Y \pm 1)$, $2 \nmid Y$, then the equation (4) has no non-trivial solutions.

(ii) Cyclotomic fields. By factoring the equation $x^l + y^l = 2z^l$ in the cyclotomic field $\boldsymbol{Q}(\zeta_l)$, Dénes [2] obtained the following:

Theorem B. *Let l be a regular prime. Suppose that the order of 2 modulo l is even or $\frac{l-1}{2}$. If $2^{l-1} \not\equiv 1 \pmod{l^2}$, then the Diophantine equation $x^l + y^l = 2z^l$ has no non-trivial integral solutions.*

It follows from Theorem B that if l is any prime < 31, then the equation (4) has no non-trivial solutions.

(iii) Diophantine approximation. Domar [3] showed the following theorem, using a result concerning Diophantine approximation by Siegel [11] whose proof depends upon properties of hypergeometric series.

Theorem C. *Let a, b be positive integers and $l \geq 5$. The equation $ax^l - by^l = \pm 1$ has at most two solutions in positive integers x, y.*

This implies that the equation (4) has at most one solution, apart from $X = 1$, $Y = 1$ (for fixed l).

(iv) Elliptic curves. Recently Darmon [1] has proved the following theorem, applying Frey's idea [5] which reduces Fermat's Last Theorem to a problem of certain elliptic curves.

Theorem D. *Let $l > 13$ be prime. If the Shimura-Taniyama conjecture is true, then the equation $x^l + y^l = z^2$ has no non-trivial solutions when $l \equiv 1 \pmod 4$ and $(x, y, z) = 1$.*

It follows from Theorem D that if the Shimura-Taniyama conjecture is true, then the equation (4) has no non-trivial solutions when $l \equiv \pmod 4$.

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