47. *The Residual Spectrum of Sp(2)*

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Introduction. Let G be the rank two symplectic group $Sp(2)$ defined over a number field k:

$$G = \left\{ g \in GL(4) \mid g \left(\begin{array}{c|c} 0 & 1_2 \\ \hline -1_2 & 0 \end{array} \right) {}^t g = \left(\begin{array}{c|c} 0 & 1_2 \\ \hline -1_2 & 0 \end{array} \right) \right\}.$$

We write $\boldsymbol{A}$ for the ring of adeles of k. We have the topological group $G(\boldsymbol{A})$, in which $G(k)$ is contained as a discrete subgroup with finite covolume. The Hilbert space $L^2(G(k)\backslash G(\boldsymbol{A}))$ of square integrable functions on the quotient space $G(k)\backslash G(\boldsymbol{A})$ is called the space of *L^2-automorphic forms*. We are interested in the right regular representation R of $G(\boldsymbol{A})$ on $L^2(G(k)\backslash G(\boldsymbol{A}))$:

$$[R(g)\phi](x) := \phi(xg), \quad (g \in G(\boldsymbol{A}), \ \phi \in L^2(G(k)\backslash G(\boldsymbol{A}))).$$

This representation decomposes into a direct sum of two $G(\boldsymbol{A})$-invariant closed subspaces $L^2_{disc}(G(k)\backslash G(\boldsymbol{A}))$ and $L^2_{cont}(G(k)\backslash G(\boldsymbol{A}))$. $L^2_{disc}(G(k)\backslash G(\boldsymbol{A}))$ is a direct sum of irreducible representations of $G(\boldsymbol{A})$, and $L^2_{cont}(G(k)\backslash G(\boldsymbol{A}))$ is a direct integral of irreducible $G(\boldsymbol{A})$-modules.

Take a k-rational parabolic subgroup $P = MU$ of G. Here M is a Levi factor of P and U is its unipotent radical. We define the constant term of $\phi \in L^2(G(k)\backslash G(\boldsymbol{A}))$ along P by

$$\phi^{(P)}(g) := \int_{U(k)\backslash U(\boldsymbol{A})} \phi(ug)\, du.$$

The closed subspace $L^2_{cusp}(G(k)\backslash G(\boldsymbol{A}))$ of *L^2-cusp forms* is spanned by those $\phi \in L^2(G(k)\backslash G(\boldsymbol{A}))$ such that $\phi^{(P)}$ vanishes almost everywhere and for all proper k-parabolic subgroup P of G. Then it is known that $L^2_{cusp}(G(k)\backslash G(\boldsymbol{A}))$ is contained in $L^2_{disc}(G(k)\backslash G(\boldsymbol{A}))$. The residual spectrum of $G(\boldsymbol{A})$ is the orthogonal complement of $L^2_{cusp}(G(k)\backslash G(\boldsymbol{A}))$ in $L^2_{disc}(G(k)\backslash G(\boldsymbol{A}))$. In this note we report on the irreducible decomposition of this residual spectrum in the case of totally real k.

§1. Preliminaries. 1.1. Notations and conventions. Let k be a totally real number field. We write $\boldsymbol{A}_\infty$ and A_f for the infinite and finite component of $\boldsymbol{A}$ respectively. $|\ |_{\boldsymbol{A}}$ denotes the idele norm of $\boldsymbol{A}^\times$. For each place v of k, we write k_v for the completion of k at v. If v is finite, $\mathcal{O}_v$ denotes the maximal compact subring of k_v.

Let $G = Sp(2)$ be as in the introduction. We fix a minimal k-parabolic subgroup P_0 of G and its Levi factor M_0. The k-split component A_0 of the center of M_0 equals M_0. Let $\Delta(P_0, A_0) = \{\alpha_1, \alpha_2\}$ is the set of simple roots of A_0 in P_0, where α_1 and α_2 denote the short and the long root respectively. Also we fix a good maximal compact subgroup $K = \Pi_v K_v$ of $G(\boldsymbol{A})$ so that we have an Iwasawa decomposition $G(\boldsymbol{A}) = P_0(\boldsymbol{A})K$. We write K_∞ for $\Pi_{v|\infty} K_v \subset G(\boldsymbol{A}_\infty)$.

We call those k-parabolic subgroups of G which contain the minimal parabolic subgroup P_0 *standard parabolic subgroups*. Each standard parabolic subgroup P has a unique Levi component M which contains M_0. The set of simple roots $\Delta(P_0 \cap M, A_0)$ of A_0 in $M \cap P_0$ is a subset of $\Delta(P_0, A_0)$. This gives a bijection between the standard parabolic subgroups of G and the subsets of $\Delta(P_0, A_0)$. We have two proper standard parabolic subgroups $P_i = M_i U_i$ $(1 \leq i \leq 2)$ of G other than $P_0 = M_0 U_0$. Each P_i is attached to $\Delta(P_0 \cap M_i, A_0) = \{\alpha_i\}$ under the above bijection.

For each standard parabolic subgroup $P = MU$, we write A_M for the k-split component of the center of M and $A_M(\boldsymbol{R})_+$ for the identity component of $A_M(\boldsymbol{R})$ in the topology of real Lie groups. $A_M(\boldsymbol{R})_+$ is diagonally embedded in $M(\boldsymbol{A}_\infty)$ and is considered as a subgroup of $M(\boldsymbol{A})$. The real Lie algebra of A_M is denoted by $\mathfrak{a}_M$ and $\mathfrak{a}^*_{M,\boldsymbol{C}}$ denotes its complexified dual space. We have the usual Harish-Chandra map $H_M : M(\boldsymbol{A}) \to \mathfrak{a}_M$. It is extended to a map from $G(\boldsymbol{A})$ by the Iwasawa decomposition fixed above:

$$H_M(g = umk) := H_M(m), \quad (u \in U(\boldsymbol{A}),\ m \in M(\boldsymbol{A}),\ k \in K).$$

Then each $\lambda \in \mathfrak{a}^*_{M,\boldsymbol{C}}$ is identified with the map $\lambda : G(\boldsymbol{A}) \ni g \to \exp\langle H_M(g), \lambda\rangle \in \boldsymbol{C}^\times$, which is restricted to a principal quasi-character of $M(\boldsymbol{A})$. Let $M(\boldsymbol{A})^1$ be the kernel of H_M in $M(\boldsymbol{A})$. The Weyl group of A_M in G is denoted by $\Omega(A_M)$. $\omega_i \in \Omega(A_0)$ denotes the simple reflection attached to $\alpha_i (i = 1, 2)$. We normalize various measures as in [1].

1.2. Pseudo-Eisenstein series. We write $\mathfrak{Z}(G(\boldsymbol{A}_\infty))$ for the center of the universal enveloping algebra of the complexified Lie algebra for the Lie group $G(\boldsymbol{A}_\infty)$. Let $P = MU$ be a standard parabolic subgroup of G. A function $\phi : U(\boldsymbol{A})M(k)\backslash G(\boldsymbol{A}) \to \boldsymbol{C}$ is said to be a *cusp form* on $U(\boldsymbol{A})M(k)\backslash G(\boldsymbol{A})$ if

(1) ϕ is of moderate growth,

(2) ϕ is smooth and K-finite on the right,

(3) ϕ is $\mathfrak{Z}(G(\boldsymbol{A}_\infty))$-finite,

(4) $\phi^{(P')}(g) := \int_{U'(k)\backslash U'(\boldsymbol{A})} \phi(ug)\, du = 0$ for any $g \in G(\boldsymbol{A})$ and any proper k-parabolic subgroup $P' = M'U'$ of P.

We write the space of cusp forms on $U(\boldsymbol{A})M(k)\backslash G(\boldsymbol{A})$ as $A_0(U(\boldsymbol{A})M(k)\backslash G(\boldsymbol{A}))$. The space $A_0(M(k)\backslash M(\boldsymbol{A}))$ of cusp forms on $M(k)\backslash M(\boldsymbol{A})$ is defined by replacing G with M in the above definition. These are considered as trivial bundles over $\mathfrak{a}^*_{M,\boldsymbol{C}}$ equipped with the $G(\boldsymbol{A}_f) \times (Lie G(\boldsymbol{A}_\infty) \otimes_{\boldsymbol{R}} \boldsymbol{C}, K_\infty)$-module structure ($(M(\boldsymbol{A}_f) \times (Lie M(\boldsymbol{A}_\infty) \otimes_{\boldsymbol{R}} \boldsymbol{C}, K_\infty \cap M(\boldsymbol{A}_\infty))$-module structure resp.) under the right translation action. Their fibers over $\lambda \in \mathfrak{a}^*_{M,\boldsymbol{C}}$ are the spaces of cusp forms ϕ such that $\phi(ag) = a^{\lambda+\rho_P}\phi(g)$ ($\phi(am) = a^\lambda\phi(m)$ resp.) for $a \in A_M(\boldsymbol{R})_+$. Here ρ_P denotes the half sum of the positive roots of A_M in P identified with an element of $\mathfrak{a}^*_M$.

It is known that every irreducible $M(\boldsymbol{A}_f) \times (Lie M(\boldsymbol{A}_\infty) \otimes_{\boldsymbol{R}} \boldsymbol{C}, K_\infty \cap M(\boldsymbol{A}_\infty))$-subquotient π of $A_0(M(k)\backslash M(\boldsymbol{A}))$ is a direct summand. Such π is called a *cuspidal automorphic representation* of $M(\mathbf{A})$. For a cuspidal automorphic representation π, we have a subbundle $\mathfrak{P} := \pi \otimes \mathfrak{a}^*_{M,\mathbf{C}}$ of $A_0(M(k)\backslash M(\mathbf{A}))$. We write $A_0(U(\mathbf{A})M(k)\backslash G(\mathbf{A}))_{\mathfrak{P}}$ for the subbundle of $A_0(U(\mathbf{A})M(k)\backslash G(\mathbf{A}))$ which consists of ϕ such that

$$\phi_k(m) := e^{\langle -\rho P, H_M(m)\rangle}\phi(mk) \quad (m \in M(\mathbf{A}))$$

belongs to $\mathfrak{P}$ for any $k \in K$. We call a pair $(M, \mathfrak{P})$ of above type a *cuspidal datum* and write $P_{(M,\mathfrak{P})}$ for the space of K-finite Paley-Wiener sections of this bundle (see [3] II.1.2).

For each $\phi \in P_{(M,\mathfrak{P})}$, its Fourier transform

$$F(\phi)(g) := \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\dim\mathfrak{a}_M}\int_{\lambda\in\sqrt{-1}\mathfrak{a}^*_M}\phi(\lambda\otimes\pi)(g)\,d\lambda$$

is independent of $\pi \in \mathfrak{P}$, and is smooth and compactly supported on $M(\mathbf{A})^1\backslash M(\mathbf{A})$. Thus the sum

$$\theta_\phi(g) := \sum_{\gamma\in P(k)\backslash G(k)} F(\phi)(\gamma g), \quad (g \in G(\mathbf{A}))$$

converges absolutely and uniformly on any compact subsets of $G(\mathbf{A})$ and defines a rapidly decreasing function on $G(k)\backslash G(\mathbf{A})$. The maps $P_{(M,\mathfrak{P})} \ni \phi \to \theta_\phi \in L^2(G(k)\backslash G(\mathbf{A}))$ are $G(\mathbf{A}_f) \times (Lie G(\mathbf{A}_\infty)\otimes_{\mathbf{R}}\mathbf{C}, K_\infty)$-equivariant, and the union of their images, where $(M, \mathfrak{P})$ runs over all cuspidal data, spans a dense subspace of $L^2(G(k)\backslash G(\mathbf{A}))$. More precisely, the closed span of these images, where $(M, \mathfrak{P})$ runs over all cuspidal data such that $M = G$, is $L^2_{cusp}(G(k)\backslash G(\mathbf{A}))$. Also the images θ_ϕ's where $(M, \mathfrak{P})$ runs over cuspidal data with $M = M_i$ for $i = 0, 1, 2$ span a dense subspace of the orthogonal complement of $L^2_{cusp}(G(k)\backslash G(\mathbf{A}))$.

§.2 The result. From above, it is enough to determine the image of maps $P_{(M_i,\mathfrak{P})} \ni \phi \to \theta_\phi \in L^2(G(k)\backslash G(\mathbf{A}))$ $(i = 0, 1, 2)$ for the study of residual spectrum. For this purpose, we analyze the L^2-scalar product of two such images θ_ϕ and $\theta_{\phi'}$. Here we assume $\phi \in P_{(M_i,\mathfrak{P})}$, $\phi' \in P_{(M_j,\mathfrak{P}')}$. The scalar product equals zero unless $i = j$ and $\mathfrak{P}$ and $\mathfrak{P}'$ are conjugate to each other under $\Omega(A_{M_i})$. In that case it is given by

$$\langle\theta_\phi, \theta_{\phi'}\rangle_{L^2(G(k)\backslash G(\mathbf{A}))} = \left(\frac{1}{2\pi\sqrt{-1}}\right)^{\dim\mathfrak{a}_M}\int_{\pi\in\mathfrak{P}, Re\pi=\lambda_0}\sum_{\omega\in\Omega(\mathfrak{P},\mathfrak{P}')}\langle M(\omega, \pi)\phi(\pi), \phi'(-\omega\bar{\pi})\rangle\,d\pi$$

(see Théorème II.2.1. in [3]). Here $M(\omega, \pi)$ is the usual intertwining operator and $\Omega(\mathfrak{P}, \mathfrak{P}') := \{\omega \in \Omega(A_{M_i}) : \omega\mathfrak{P} = \mathfrak{P}'\}$. Also $\lambda_0 \in \mathfrak{a}^*_M$ is such that $\langle\lambda_0 - \rho_P, \alpha^\vee\rangle > 0$ for all $\alpha \in \Delta(P_i, A_{M_i})$. As for other notations, see Chapter II of [3].

It is known that the integrand in the above formula is a meromorphic function on $\mathfrak{P}$. Thus, by applying the residue theorem, the above formula equals the sum of the integral with the same integrand but over $\{\pi \in \mathfrak{P}; Re\pi = 0\}$ and contributions of the residues of the integrand. Some contributions of the residues are still integrals and represents the scalar product of $L^2_{cont}(G(k)\backslash G(\mathbf{A}))$-components of θ_ϕ and $\theta_{\phi'}$ together with the first integral. But the other contributions are discrete and these give the scalar product formula for the residual spectrum. Now to describe our results we recall certain automorphic representations of $G(\boldsymbol{A})$ attached to quadratic forms from [2].

Let $(V, \langle,\rangle)$ be a two dimensional quadratic space defined over k. We write $O(V)$ for the orthogonal group of $(V, \langle,\rangle)$. We fix a non-trivial character $\psi = \otimes_v \psi_v$ of $\boldsymbol{A}/k$. At each place v of k, we have a dual reductive pair $(Sp(2, k_v), O(V, k_v))$ and its oscillator representation ω_{ψ_v} on $\mathcal{S}(V_v^2)$. Here $V_v := V \otimes_k k_v$ and $\mathcal{S}(V_v^2)$ is the Schwartz space on $V_v^{\oplus 2}$. Then the θ-lift $R(V_v)$ of the trivial representation of $O(V, k_v)$ is the representation of $Sp(2, k_v)$ on the $O(V, k_v)$-coinvariant space of $\mathcal{S}(V_v^2)$. This is known to be irreducible. Moreover if v is finite, ψ_v is of order 0 and if $\Delta(V_v) := -\det(V_v, \langle,\rangle) \in \mathcal{O}_v$, then $R(V_v)$ is spherical. Thus we have an irreducible smooth representation of $Sp(2, \boldsymbol{A})$ on the restricted tensor product space $\otimes_v R(V_v)$. We write $R(V)$ for this representation. Now we have the following results.

Theorem. *The residual spectrum of the rank two symplectic group $G = Sp(2)$ over a totally real number field is a direct sum of the following representations. Each occurs with multiplicity one.*

(1) *The trivial representation $\mathbf{1}_{G(\boldsymbol{A})}$.*

(2) *The θ-lifts $R(V)$ of trivial representations of the orthogonal groups $O(V, \boldsymbol{A})$ for two dimensional non-hyperbolic quadratic spaces V over k.*

(3) *The unique irreducible quotients of global parabolically induced representations $\mathrm{Ind}_{P_1(\boldsymbol{A})}^{G(\boldsymbol{A})}[\mathfrak{S}(P_1) \otimes \mathbf{1}_{U_1(\boldsymbol{A})}]$ where $\mathfrak{S}(P_1)$ runs over cuspidal automorphic representations of $M_1(\boldsymbol{A}) \simeq GL(2, \boldsymbol{A})$ whose central characters equal $|\ |_{\boldsymbol{A}}$ and their standard L-functions $L(s, \mathfrak{S}(P_1))$ do not vanish at $s = 0$.*

(4) *The unique irreducible quotients of $\mathrm{Ind}_{P_2(\boldsymbol{A})}^{G(\boldsymbol{A})}[\mathfrak{S}(P_2) \otimes \mathbf{1}_{U_2(\boldsymbol{A})}]$ where $\mathfrak{S}(P_2) = \chi \otimes \sigma$ runs over cuspidal automorphic representations of $M_2(\boldsymbol{A})$ such that*

$$\chi = |\ |_{\boldsymbol{A}} \eta_{F/k}, \quad \sigma \hookrightarrow \pi(\Omega)|_{SL(2,\boldsymbol{A})}$$

for some quadratic extension F/k and a character Ω of $\boldsymbol{A}_F^\times/F^\times$.

Here in (4), *$\pi(\Omega)$ is the automorphic representation of $GL(2, \boldsymbol{A})$ attached to Ω by the Weil lifting and $\eta_{F/k}$ is the quadratic character of $\boldsymbol{A}^\times/k^\times$ which corresponds to F/k by the classfield theory.*

References

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