46. On Representations of Finite Groups in the Space of Modular Forms of Half-integral Weight

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Introduction. Let p be a prime and k an integer. In 1940, Hecke studied a representation of $SL_2(\boldsymbol{Z}/p\boldsymbol{Z})$ which is realized in the space of modular forms of level p and of weight k and obtained beautiful results.

In this paper, we study a similar representation $\pi_{k+1/2}$ which is realized in the space of cusp forms of level $4p$ and of weight $k+1/2$. In particular, we study in detail the subrepresentation ρ_f generated by Hecke common eigenform f of level $4p$ ("newform"). Then we have some completely different facts from the results in the case of integral weight. For example, ρ_f is always irreducible, and if f is of Neben-type, whether ρ_f is "residual" or "non-residual" (cf. below (1.2)) is determined by the Atkin-Lehner involution $W(p)$ (cf. Theorem (4.1) for the details).

Finally, we remark that the class number of $\boldsymbol{Q}(\sqrt{-p})$ also occurs in our results as in the classical work of Hecke (cf. Remark(4.2)).

§0 Preliminaries. Throughout this paper, we keep to the notation in [4]. In particular, we use the following general notation.

Let k denote a positive integer and p an odd prime number. If $z \in \boldsymbol{C}$ and $x \in \boldsymbol{C}$, we put $z^x = \exp(x \cdot \log(z))$ with $\log(z) = \log(|z|) + \sqrt{-1}\arg(z)$, $\arg(z)$ being determined by $-\pi < \arg(z) \leq \pi$. Also we put $\boldsymbol{e}(z) = \exp(2\pi\sqrt{-1}z)$.

Let $\mathfrak{H}$ be the complex upper half plane. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ and $z \in \mathfrak{H}$, we define function $j(\gamma, z)$ on $\mathfrak{H}$ by: $j(\gamma, z) = \left(\frac{-1}{d}\right)^{-1/2}\left(\frac{c}{d}\right)(cz+d)^{1/2}$. Let $\mathfrak{G}(k+1/2)$ be the group consisting of pairs (α, φ), where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\boldsymbol{R})$ and φ is a holomorphic function on $\mathfrak{H}$ satisfying $\varphi(z) = t(\det\alpha)^{-k/2-1/4}(cz+d)^{k+1/2}$ with $t \in \boldsymbol{C}$ and $|t| = 1$. The group law is defined by: $(\alpha, \varphi(z)) \cdot (\beta, \psi(z)) = (\alpha\beta, \varphi(\beta z)\psi(z))$. For a complex-valued function f on $\mathfrak{H}$ and $(\alpha, \varphi) \in \mathfrak{G}(k+1/2)$, we define a function $f|(\alpha, \varphi)$ on $\mathfrak{H}$ by: $f|(\alpha, \varphi)(z) = \varphi(z)^{-1}f(\alpha z)$.

§1. For a positive integer N, we put $\boldsymbol{G}(N) := SL_2(\boldsymbol{Z}/N\boldsymbol{Z})$, $\boldsymbol{B}(N) := \left\{\begin{pmatrix} a^{-1} & b \\ 0 & a \end{pmatrix} \in \boldsymbol{G}(N)\right\}$, $\boldsymbol{U}(N) := \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \boldsymbol{B}(N)\right\}$.

Denote by $\mathcal{L}$ the lifting $\Gamma_0(4) \ni \gamma \mapsto \gamma^* = (\gamma, j(\gamma, z)^{2k+1})$. Then we put for an odd prime p, $\Delta(4p) := \mathcal{L}(\Gamma(4p))$, $\Delta_1(4p) := \mathcal{L}(\Gamma_1(4p))$, and $\Delta_0(4p) := \mathcal{L}(\Gamma_0(4p))$. Moreover, by $S(k+1/2, \Delta(4p))$, we denote the space of all cusp forms of weight $k+1/2$ with respect to $\Delta(4p)$.

For an even character χ modulo $4p$, define a subspace of $S(k+1/2, \Delta(4p))$ by:

$$S(k+1/2, 4p, \chi) := \left\{ \begin{array}{l} f \in S(k+1/2, \Delta(4p))\ ; f \mid \gamma^{*} = \chi(d) f \\ \text{for any } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4p) \end{array} \right\}.$$

Since $S(k+1/2, 4p, \chi) = \{0\}$ for each odd character χ, we consider only even characters.

Since $\Delta(4p)$ is a normal subgroup of $\Delta_0(4) := \mathcal{L}(\Gamma_0(4))$, we get a representation $\pi_{k+1/2}$ of $\Delta_0(4)/\Delta(4p) \cong \boldsymbol{B}(4) \times \boldsymbol{G}(p)$ on $S(k+1/2, \Delta(4p))$ defined by:

$$[\pi_{k+1/2}(\gamma \bmod 4p)] f = f \mid \gamma^{*-1},\ f \in S(k+1/2, \Delta(4p)),\ \gamma \in \Gamma_0(4).$$

For $f \in S(k+1/2, \Delta(4p))$, let ρ_f denote the subrepresentation of $\pi_{k+1/2}$ generated by f, i.e., we set $\rho_f := \langle f \mid \gamma^{*}\ ;\ \gamma \in \Gamma_0(4) \rangle_{\boldsymbol{C}}$.

For any non-zero $f \in S(k+1/2, 4p, \chi)$, we study this representation ρ_f. By restricting $\pi_{k+1/2}$, $\boldsymbol{C}f$ is a one-dimensional representation space of $\boldsymbol{B}(4) \times \boldsymbol{B}(p) \cong \Delta_0(4p)/\Delta(4p)$. We denote this representation by $\underline{\underline{\chi}}$. We also set the representations $\underline{\underline{\chi_2}}$ and $\underline{\underline{\chi_p}}$ by:

$$\underline{\underline{\chi_2}} : \boldsymbol{B}(4) \ni \begin{pmatrix} a^{-1} & b \\ 0 & a \end{pmatrix} \mapsto \chi_2(a)^{-1},\ \underline{\underline{\chi_p}} : \boldsymbol{B}(p) \ni \begin{pmatrix} a^{-1} & b \\ 0 & a \end{pmatrix} \mapsto \chi_p(a)^{-1},$$

where χ_2 (resp. χ_p) is the 2 (resp. p)-primary component of χ. Then $\underline{\underline{\chi}} = \underline{\underline{\chi_2}} \otimes \underline{\underline{\chi_p}}$.

Moreover we can define a surjective homomorphism between $\boldsymbol{B}(4) \times \boldsymbol{G}(p)$-modules: $\mathrm{Ind}_{\boldsymbol{B}(4)\times\boldsymbol{B}(p)}^{\boldsymbol{B}(4)\times\boldsymbol{G}(p)} \underline{\underline{\chi}} = \underline{\underline{\chi_2}} \otimes \mathrm{Ind}_{\boldsymbol{B}(p)}^{\boldsymbol{G}(p)} \underline{\underline{\chi_p}} \to \rho_f$ by $\sum_\xi a_\xi \xi \otimes f \mapsto \sum_\xi a_\xi \pi_{k+1/2}(\xi) f$ ($a_\xi \in \boldsymbol{C}$, $\xi \in (\boldsymbol{B}(4) \times \boldsymbol{G}(p))/(\boldsymbol{B}(4) \times \boldsymbol{B}(p))$). From this, we can identify ρ_f with a $\boldsymbol{B}(4) \times \boldsymbol{G}(p)$-submodule of $\underline{\underline{\chi_2}} \otimes \mathrm{Ind}_{\boldsymbol{B}(p)}^{\boldsymbol{G}(p)} \underline{\underline{\chi_p}}$.

As to the representation $\underline{\underline{\chi_2}} \otimes \mathrm{Ind}_{\boldsymbol{B}(p)}^{\boldsymbol{G}(p)} \underline{\underline{\chi_p}}$, the following assertion is well-known.

Proposition (1.1) ([3, Chapter 7, pp. 54-60]). (1) *If* $\chi^2 \neq \mathbf{1} (\Leftrightarrow \chi_p^{\ 2} \neq \mathbf{1})$, $\underline{\underline{\chi_2}} \otimes \mathrm{Ind}_{\boldsymbol{B}(p)}^{\boldsymbol{G}(p)} \underline{\underline{\chi_p}}$ *is an irreducible representation.*

(2) *If* $\chi = \mathbf{1}$ (*the trivial representation*),

$$\mathbf{1} \otimes \mathrm{Ind}_{\boldsymbol{B}(p)}^{\boldsymbol{G}(p)} \mathbf{1} = (\mathbf{1} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathfrak{S}_p).$$

Here, $\mathfrak{S}_p$ *is an irreducible representation of* $\boldsymbol{G}(p)$ *of degree* p *which is called Steinberg representation.*

(3) *If* $\chi = \left(\frac{p}{\cdot}\right)$ (*Kronecker symbol*),

$$\underline{\underline{\chi_2}} \otimes \mathrm{Ind}_{\boldsymbol{B}(p)}^{\boldsymbol{G}(p)} \underline{\underline{\chi_p}} = (\underline{\underline{\chi_2}} \otimes \mathfrak{S}_{(p+1)/2}) \oplus (\underline{\underline{\chi_2}} \otimes \mathfrak{S}'_{(p+1)/2}).$$

Here $\mathfrak{S}_{(p+1)/2}$ *and* $\mathfrak{S}'_{(p+1)/2}$ *denote irreducible representations of* $\boldsymbol{G}(p)$ *of degree* $(p+1)/2$, *which are not equivalent to each other and satisfy the following*:

(1.2)

$$\mathfrak{S}_{(p+1)/2} \mid \boldsymbol{U}(p) \cong \psi_0 \oplus (\oplus_{a \in \boldsymbol{F}_p^{\times 2}} \psi_a),\quad \mathfrak{S}'_{(p+1)/2} \mid \boldsymbol{U}(p) \cong \psi_0 \oplus (\oplus_{a \in \boldsymbol{F}_p^{\times} - \boldsymbol{F}_p^{\times 2}} \psi_a),$$

where for any $a \in \boldsymbol{F}_p$, *we define* $\psi_a \in \widehat{\boldsymbol{U}(p)}$ *by*: $\psi_a\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) = \psi(au)$ *and* $\psi(x \bmod p) = \boldsymbol{e}(x/p)$ $(x \in \boldsymbol{Z})$. *We call* $\mathfrak{S}_{(p+1)/2}$ (*resp.* $\mathfrak{S}'_{(p+1)/2}$) *the residual* (*resp. non-residual*) *representation.*

Corollary (1.3). *For any non-zero* $f \in S(k+1/2, 4p, \chi)$,

$$\rho_f \cong \begin{cases} \underline{\underline{\chi_2}} \otimes \mathrm{Ind}_{\boldsymbol{B}(p)}^{\boldsymbol{G}(p)} \underline{\underline{\chi_p}}, & \text{if } \chi^2 \neq \mathbf{1}, \\ \mathbf{1} \otimes \mathbf{1},\ \mathbf{1} \otimes \mathfrak{C}_p,\ (\mathbf{1} \otimes \mathbf{1}) \oplus (\mathbf{1} \otimes \mathfrak{C}_p), & \text{if } \chi = \mathbf{1}, \\ \underline{\underline{\chi_2}} \otimes \mathfrak{C}_{(p+1)/2},\ \underline{\underline{\chi_2}} \otimes \mathfrak{C}'_{(p+1)/2},\ (\underline{\underline{\chi_2}} \otimes \mathfrak{C}_{(p+1)/2}) \oplus (\underline{\underline{\chi_2}} \otimes \mathfrak{C}'_{(p+1)/2}), & \text{if } \chi = \left(\frac{p}{\ }\right). \end{cases}$$

§2. Now, we shall study the cases of $\chi = \mathbf{1}$ and $\left(\frac{p}{\ }\right)$ in detail. From now on until the end of this paper, we fix $\chi =$ either $\mathbf{1}$ or $\left(\frac{p}{\ }\right)$.

For any $a \in \boldsymbol{F}_p$, take $\zeta_a \in SL_2(\boldsymbol{Z})$ such that

$$\zeta_a \equiv \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 4 \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \pmod p. \end{cases}$$

The set $\{\zeta_a^* \mid a \in \boldsymbol{F}_p\} \cup \left\{\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1\right)\right\}$ gives a complete system of representatives of $\Delta_0(4p) \backslash \Delta_0(4)$. Then define an operator X^* on $S(k+1/2, \Delta(4p))$ by: $f \mid X^* := \sum_{a \in \boldsymbol{F}_p} f \mid \zeta_a^*$.

Proposition (2.1). *We assume that* $\chi =$ *either* $\mathbf{1}$ *or* $\left(\frac{p}{\ }\right)$. *Put* $\mathfrak{g}_p = \sqrt{\left(\frac{-1}{p}\right) p}$. *For a non-zero* $f \in S(k+1/2, 4p, \chi)$, *the following hold.*

(1) X^* *induces an operator on* $S(k+1/2, 4p, \chi)$.

(2) $f \mid X^{*2} = \begin{cases} (p-1) f \mid X^* + pf, & \text{if } \chi = \mathbf{1}, \\ \left(\frac{-1}{p}\right) pf, & \text{if } \chi = \left(\frac{p}{\ }\right). \end{cases}$

(3) ρ_f *is irreducible* $\Leftrightarrow f$ *is an eigenform of* X^*.

(4) *Let* $\chi = \mathbf{1}$. *Then*

$$\begin{cases} \rho_f \cong \mathbf{1} \otimes \mathbf{1} & \Leftrightarrow\ f \mid X^* = pf, \\ \rho_f \cong \mathbf{1} \otimes \mathfrak{C}_p & \Leftrightarrow\ f \mid X^* = -f. \end{cases}$$

(5) *Let* $\chi = \left(\frac{p}{\ }\right)$. *Then*

$$\begin{cases} \rho_f \cong \underline{\underline{\chi_2}} \otimes \mathfrak{C}_{(p+1)/2} & \Leftrightarrow\ f \mid X^* = \left(\frac{-1}{p}\right) \mathfrak{g}_p f, \\ \rho_f \cong \underline{\underline{\chi_2}} \otimes \mathfrak{C}'_{(p+1)/2} & \Leftrightarrow\ f \mid X^* = -\left(\frac{-1}{p}\right) \mathfrak{g}_p f. \end{cases}$$

Proof. (1) From easy computation, we have $f \mid X^* \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^* = f \mid X^*$. Since $\Delta_1(4p) = \left\langle \Delta(4p), \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^* \right\rangle$, $f \mid X^* \in S(k+1/2, \Delta_1(4p))$. The assertion follows from checking the action of any element $\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}\right) \in \boldsymbol{B}(4) \times \boldsymbol{B}(p)$ on $f \mid X^*$.

(2) $f \mid X^{*2} = \sum_{a,b \in \boldsymbol{F}_p} f \mid \zeta_a^* \zeta_b^*$. We divide the right-hand side into two parts S_1 (the part of $a = 0$) and S_2 (the part of $a \neq 0$). Then from easy computation, $S_1 = p\chi_p(-1) f$ and $S_2 = (p-1) f \mid X^*$ or 0 according to $\chi = \mathbf{1}$

or $\left(\frac{p}{-}\right)$.

(3) Assume that ρ_f is irreducible. Then $1 = \langle \rho_f, \underline{\underline{\chi_2}} \otimes \mathrm{Ind}_{B(p)}^{G(p)} \underline{\underline{\chi_p}} \rangle_{B(4)\times G(p)} = \langle \rho_f |_{B(4)\times B(p)}, \underline{\underline{\chi}} \rangle_{B(4)\times B(p)}$. From (1), we have $g := f | X^* \in S(k+1/2, 4p, \chi) \cap \rho_f$. If $g \neq 0$, both $\boldsymbol{C}f$ and $\boldsymbol{C}g$ give the subrepresentation $\underline{\underline{\chi}}$ of $\rho_f |_{B(4)\times B(p)}$. Hence $\boldsymbol{C}f = \boldsymbol{C}g$. Next, we assume that $f | X^* = \lambda f (\lambda \in \boldsymbol{C})$. Then $\dim \rho_f = \dim \langle f, f | \zeta_a^* ; a \in \boldsymbol{F}_p \rangle_{\boldsymbol{C}} \leq p$. From this and Corollary (1.3), ρ_f is irreducible.

(4) Let $\rho_f \cong \mathbf{1} \otimes \mathfrak{C}_p$. $g = f + f | X^* = \sum_{\gamma^* \in \Delta_0(4p) \backslash \Delta_0(4)} f | \gamma^*$ is $\Delta_0(4)$-invariant. Hence if $g \neq 0$, $\boldsymbol{C}g$ is a $\boldsymbol{B}(4) \times \boldsymbol{G}(p)$-submodule of ρ_f which is isomorphic to $\mathbf{1} \otimes \mathbf{1}$. Therefore we have $g = 0$. The assertion for $\mathbf{1} \otimes \mathbf{1}$ is trivial.

The contrary easily follows form the above, (3), and Corollary (1.3).

(5) For any $u \in \boldsymbol{F}_p$, put $f_u := \sum_{a \in \boldsymbol{F}_p} \boldsymbol{e}(-ua/p) f | \zeta_a^* \in \rho_f$. From similar computation to (2), $f_u | X^* = \left(\frac{-1}{p}\right) pf + \left(\frac{u}{p}\right) \mathfrak{g}_p f | X^*$. If $f_u \neq 0$, $\boldsymbol{C}f_u$ gives the subrepresentation $\underline{\underline{\chi_2}} \otimes \psi_{(-u)}$ of $\rho_f |_{B(4)\times U(p)}$.

Let $\rho_f \cong \underline{\underline{\chi_2}} \otimes \mathfrak{C}_{(p+1)/2}$. Then $f | X^* = \lambda f (\lambda \in \boldsymbol{C})$ by (3). Take u such that $\left(\frac{-u}{p}\right) = -1$. From the condition (1.2), we have $f_u = 0$. Hence $0 = f_u | X^* = \left(\frac{-1}{p}\right) (p - \mathfrak{g}_p \lambda) f$. Therefore $\lambda = \left(\frac{-1}{p}\right) \mathfrak{g}_p$. As to $\underline{\underline{\chi_2}} \otimes \mathfrak{C}'_{(p+1)/2}$, we can verify in the same way. The contrary is easily shown from the above results and Corollary (1.3).

§3. Now, we shall characterize irreducibility of ρ_f in terms of Fourier coefficients of f. We introduce the operators $U(p)$, $\tilde{W}(p)$, Y_p, and Hecke operator $\tilde{T}(n^2) = \tilde{T}_{k+1/2,4p,\chi}(n^2)$ from [4]. See [4, §0 and §1] for the definitions of these operators.

Let $f \in S(k+1/2, 4p, \chi)$. Since $f | \tilde{W}(p) = f | \zeta_0^{*-1} \left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p^{-k/2-1/4} \right)$ and $f | U(p) = p^{k/2-3/4} \sum_{a \in \boldsymbol{F}_p} f | \left(\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, p^{k/2+1/4} \right) \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^*$ ([4, pp. 151-152]), we have $f | X^* = p^{-k/2+3/4} \chi_p(-1) f | \tilde{W}(p) U(p)$ and $f | U(p) X^* = \chi_p(-1) \left(\frac{-1}{p}\right) f | Y_p U(p)$.

Put $g := f | U(p)$. Observing that the map $f \mapsto f | U(p)$ gives an isomorphism from $S(k+1/2, 4p, \chi)$ onto $S\left(k+1/2, 4p, \chi\left(\frac{p}{-}\right)\right)$ ([4, Proposition (1.28)]), $g | X^* = \lambda g \ (\lambda \in \boldsymbol{C}) \Leftrightarrow f | Y_p = \chi_p(-1) \left(\frac{-1}{p}\right) \lambda f$.

If $\chi = \mathbf{1}$, $g = f | U(p) \in S\left(k+1/2, 4p, \left(\frac{p}{-}\right)\right)$ and $\lambda = \pm \left(\frac{-1}{p}\right) \mathfrak{g}_p$. Then the following follows from [4, Proposition (1.29)].

Theorem (3.1). *Let* $(0 \neq) f = \sum_{n \geq 1} a(n) \boldsymbol{e}(nz) \in S(k+1/2, 4p, \mathbf{1})$. *Then* $f | U(p) = \sum_{n \geq 1} a(pn) \boldsymbol{e}(nz)$ *and*

$$\begin{cases} \rho_{f|U(p)} \cong \underline{\theta} \otimes \mathfrak{C}_{(p+1)/2} \Leftrightarrow a(n) = 0 \ \textit{if} \left(\frac{-n}{p}\right) = -1, \\ \rho_{f|U(p)} \cong \underline{\theta} \otimes \mathfrak{C}'_{(p+1)/2} \Leftrightarrow a(n) = 0 \ \textit{if} \left(\frac{-n}{p}\right) = +1. \end{cases}$$

Here, θ is the 2-primary component of Kronecker symbol $\left(\frac{p}{\ }\right)$.

§4. In the case of integral weight, a Hecke eigenform f with $\rho_f \cong \mathfrak{C}_{(p+1)/2}$ is very special, in fact, such f corresponds to a Grössencharacter of $\boldsymbol{Q}(\sqrt{-p})$. In our case, a Hecke eigenform f with $\rho_f \cong \underline{\underline{\chi_2}} \otimes \mathfrak{C}_{(p+1)/2}$ or $\underline{\underline{\chi_2}} \otimes \mathfrak{C}'_{(p+1)/2}$ is also a little special in the following sense.

We introduce the following subspace which is called Kohnen space.

$$S(k+1/2, 4p, \chi)_K := \begin{Bmatrix} f = \sum_{n\geq 1} a(n)e(nz) \in S(k+1/2, 4p, \chi) \ ; \\ a(n) = 0 \text{ if } \chi_2(-1)(-1)^k n \equiv 2,3 \ (\mathrm{mod}\ 4) \end{Bmatrix}.$$

We recall that we have a theory of newforms for Kohnen spaces (cf. [1], [4]).

Let $\mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4p, \chi)_K$ be the space of newforms. See [4, §3] for the defintion. The space is denoted by $S^{\mathrm{new}}_{k+1/2}(p, \chi_p)$ in [1]. From [4,§3], we know the following: $S(k+1/2, 4p, \chi)_K = \mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4p, \chi)_K \oplus \mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4, \chi)_K \oplus \mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4, \chi)_K \mid U(p^2)$; Both $\mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4p, \chi)_K$ and $\mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4, \chi)_K$ have $\boldsymbol{C}$-basis consisting of common eigenforms for all $\tilde{T}(l^2)$ (l : prime, $l \neq p$); $\mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4p, \chi)_K$ and $\mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4, \chi)_K \oplus \mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4, \chi)_K \mid U(p^2)$ correspond to the spaces $S^0(2k, p)$ and $S(2k, 1)$ respectively via Shimura correspondence; $\mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4p, \chi)_K$ is stable under the operators $U(p^2)$ and $\tilde{T}(n^2)$ with $(n, p) = 1$ ([4, Theorem (3.9-10)]). We also claim that X^* and Y_p fix the space $\mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4p, \chi)_K$. This follows from [4, Theorem (3.10-11), Propositions (1.20) and (1.28)] and [2, Theorem 4.6.19].

Theorem (4.1). *Let $(0 \neq) f \in \mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4p, \chi)_K$ be a common eigenform for all $\tilde{T}(l^2)$ (l : prime, $l \neq p$). Then we have the following.*

(1) *ρ_f is always irreducible.*

(2) *If $\chi = \mathbf{1}$, then $\rho_f \cong \mathbf{1} \otimes \mathfrak{C}_p$.*

(3) *If $\chi = \left(\frac{p}{\ }\right)$, then*

$$\begin{cases} \rho_f \cong \underline{\underline{\chi_2}} \otimes \mathfrak{C}_{(p+1)/2} \Leftrightarrow G \mid W(p) = \left(\frac{-1}{p}\right)^{k-1} G ; \\ \rho_f \cong \underline{\underline{\chi_2}} \otimes \mathfrak{C}'_{(p+1)/2} \Leftrightarrow G \mid W(p) = -\left(\frac{-1}{p}\right)^{k-1} G. \end{cases}$$

Here, $W(p)$ is the Atkin-Lehner involution on $S(2k, p)$ (see [4, p. 5]) and G is the primitive form $\in S^0(2k, p)$ which corresponds to $f \mid U(p)^{-1} =: g$ in the sense of [4, Theorem (3.11)(1)] (via Shimura Correspondence).

Proof. (1) X^* commutes with all Hecke operators $\tilde{T}(n^2)$, $(n, 2p) = 1$ ([4, Proposition (1.20)]). Then from the strong multiplicity one theorem ([4, Theorem 3.11)]), f is also an eigenform of X^* and hence ρ_f is irreducible.

(2) Suppose that $\rho_f \cong \mathbf{1} \otimes \mathbf{1}$. Then $f \mid \gamma^* = f$ for all $\gamma \in \Gamma_0(4)$. Since $S(k+1/2, 4, \mathbf{1}) \cap S(k+1/2, 4p, \mathbf{1})_K = S(k+1/2, 4, \mathbf{1})_K = \mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4, \mathbf{1})_K$, $f \in \mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4, \mathbf{1})_K \cap \mathfrak{S}^{\emptyset,\kappa}(k+1/2, 4p, \mathbf{1})_K = \{0\}$. This

is a contradiction.

(3) From [4, (3.3), Propositions (1.20) and (3.8), Theorem (3.11)], we can show that $g := f \mid U(p)^{-1} \in \mathfrak{S}^{\emptyset,\chi}(k+1/2, 4p, \mathbf{1})_K$, g is a common eigenform for all $\tilde{T}(l^2)$ (l : prime $\neq p$), and $g \mid U(p^2) = \lambda_p g$, $\lambda_p = \pm p^{k-1}$. Moreover, we defined the involution $\boldsymbol{w}_p$ on $\mathfrak{S}^{\emptyset,\chi}(k+1/2, 4p, \mathbf{1})_K$ by $g \mid \boldsymbol{w}_p = p^{-1/2}\left(\frac{-1}{p}\right)^{k+1/2} g \mid Y_p$ (cf. [4, (3.6) and Theorem (3.9)]).

This involution corresponds to the Atkin-Lehner involution $W(p)$ as follows. Take the primitive form $G \in S^0(2k, p)$ as in the above statement. Then from [4, Theorem (3.9)], we can write $\sigma_p g = -p^{1-k} g \mid U(p^2) = g \mid \boldsymbol{w}_p$, $\sigma_p := -p^{1-k}\lambda_p = \pm 1$. By using [4, Theorem (3.11)(1)] and [2, Corollary 4.6.18(2)], $\sigma_p G = -p^{1-k} G \mid U(p) = G \mid W(p)$.

Therefore, $f \mid X^* = \left(\frac{-1}{p}\right) \mathfrak{g}_p f \Leftrightarrow g \mid Y_p = \mathfrak{g}_p g \Leftrightarrow g \mid \boldsymbol{w}_p = \left(\frac{-1}{p}\right)^{k-1} g$ $\Leftrightarrow G \mid W(p) = \left(\frac{-1}{p}\right)^{k-1} G$.

Remark (4.2). For $\chi = \left(\frac{p}{\ }\right)$, take a $\boldsymbol{C}$-basis $\{f_i\}$ of $\mathfrak{S}^{\emptyset,\chi}(k+1/2, 4p, \chi)_K$ consisting of common eigenforms for all $\tilde{T}(l^2)$ (l : prime, $l \neq p$). Put $\rho_i := \rho_{f_i}$. Then we have $\boldsymbol{D} := \#\{i \mid \rho_i \cong \underline{\underline{\chi_2}} \otimes \mathfrak{C}_{(p+1)/2}\} - \#\{i \mid \rho_i \cong \underline{\underline{\chi_2}} \otimes \mathfrak{C}'_{(p+1)/2}\} = \left(\frac{-1}{p}\right)^{k-1} \operatorname{tr}(W(p) ; S(2k, p))$. In particular, when $p \geq 5$ and $k \geq 2$, we have

$$\boldsymbol{D} = \left(\frac{-1}{p}\right)^{k-1} ((-1)^k/2) \times \begin{cases} h(-4p), & \text{if } p \equiv 1 \pmod 4, \\ 4h(-p), & \text{if } p \equiv 3 \pmod 8, \\ 2h(-p), & \text{if } p \equiv 7 \pmod 8. \end{cases}$$

Here, $h(u)$ is the class number of $\boldsymbol{Q}(\sqrt{u})$.

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