40. *A Note on Base Point Free Theorem*

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(Communicated by Heisuke HIRONAKA, M. J. A., June 7, 1994)

The purpose of this note is to outline our recent results concerning Base Point Free Theorem. Details will be published elsewhere.

Let X be a non-singular projective variety with $\dim X = n$ over $\boldsymbol{C}$. And let $\Delta = \sum_{i=1}^{s} \Delta_i$ be a reduced divisor on X with only simple normal crossings.

Let $\mathbf{Strata}(\Delta) := \{\Gamma \mid 1 \leqq k \leqq n,\ 1 \leqq i_1 < i_2 < \cdots < i_k \leqq s,\ \Gamma$ is an irreducible component of $\Delta_{i_1} \cap \Delta_{i_2} \cap \cdots \cap \Delta_{i_k} \neq \phi\}$. A divisor R on (X, Δ) is *nef and log big,* if R is nef and big and $R|_\Gamma$ is nef and big for any member Γ of $\mathbf{Strata}(\Delta)$ (due to Reid [8]).

§1. Known results. Theorem 1 (Kawamata-Shokurov [4]). *If L is a nef divisor on X and $aL - (K_X + \Delta)$ is ample for some $a \geqq 0$, then $|lL|$ is base point free for $l \gg 0$.*

Theorem 2 (Kollár [5]). *Notation as in Theorem* 1. *There is a natural number l_0 depending only on n and a such that $|l_0L|$ is base point free.*

Proof. For $0 < d \ll 1$, $aL - (K_X + (1-d)\Delta)$ is ample and $(X, (1-d)\Delta)$ is Kawamata-log-terminal. **Q.E.D.**

Theorem 3 (Reid [8]). *If L is a nef divisor on X and $aL - (K_X + \Delta)$ is nef and log big on (X, Δ) for some $a \geqq 0$, then $|lL|$ is base point free for $l \gg 0$.*

Remark. Theorem 4 and the method of Kawamata [3, Lemma 2] imply the theorem above.

§2. Norimatsu type vanishing and main theorem. Theorem 4 (cf. Ein-Lazarsfeld [1] and Norimatsu [7]). *Let R be a nef and log big divisor on (X, Δ). Then $H^i(X, \mathcal{O}_X(K_X + \Delta + R)) = 0$ for $i > 0$.*

Sketchy proof. The assertion follows from the following exact sequence:

$$0 \to \mathcal{O}_X(K_X + \sum_{j<s} \Delta_j + R) \to \mathcal{O}_X(K_X + \Delta + R)$$
$$\to \mathcal{O}_{\Delta_s}(K_{\Delta_s} + \sum_{j<s} \Delta_j|_{\Delta_s} + R|_{\Delta_s}) \to 0. \qquad \textbf{Q.E.D.}$$

Main theorem. *Notation as in Theorem* 3. *There exists a natural number $f(n, a)$ (which is $\geqq a$) which depends only on n and a such that $|f(n, a)L|$ is base point free.*

Sketchy proof (Using Kawamata-Shokurov-Kollár's method [5]). We prove the theorem by induction on n.

Let $g : X \to S$ be the morphism defined by the linear system $|lL|$ for $l \gg 0$ (Theorem 3). There exists a Cartier divisor L_S on S such that $L = g^*L_S$.

We may assume that $\Delta \neq 0$ (Theorem 2).

Put $m := f(n-1, a)$. We consider the exact sequence

$$0 \to \mathcal{O}_X(mL - \Delta_i) \to \mathcal{O}_X(mL) \to \mathcal{O}_{\Delta_i}(mL) \to 0$$

for all $1 \leqq i \leqq s$. By Theorem 4, $H^1(X, \mathcal{O}_X(mL - \Delta_i)) = 0$ because $mL - \Delta_i - (K_X + \sum_{j \neq i} \Delta_j)$ is nef and log big on $(X, \sum_{j \neq i} \Delta_j)$. By the induction hypothesis, $\mathrm{Bs}\,|mL|_{\Delta_i}| = \phi$, because $aL|_{\Delta_i} - (K_{\Delta_i} + \sum_{j \neq i} \Delta_j|_{\Delta_i})$ is nef and log big on $(\Delta_i, \sum_{j \neq i} \Delta_j|_{\Delta_i})$. Thus $\mathrm{Bs}\,|mL| \cap \Delta = \phi$.

Let Z_S be an irreducible component of $\mathrm{Bs}\,|mL_S|$. Let $k = \mathrm{codim}(Z_S, S)$. Taking general elements $B_i \in |mL|$, put $B = (1/2m)B_0 + B_1 + \cdots + B_k$. Then $(X, \Delta + B)$ is log canonical outside $\mathrm{Bs}\,|mL|$ and $(X, \Delta + B)$ is not log canonical at the points belonging to the inverse image of the generic point of Z_S by g. Let $M_0 := aL - (K_X + \Delta) + (1/2)L$.

Take a log resolution $f : Y \to X$. Let

$K_Y \equiv f^*(K_X + \Delta) + \sum e_i E_i \quad (e_i \geqq -1)$;

$f^*B \equiv \sum b_i E_i$;

$f^*M_0 \equiv A + \sum p_i E_i$ (A is an ample $\boldsymbol{Q}$-divisor and $0 \leqq p_i \ll 1$).

Put $c = \min\{(e_i + 1 - p_i)/b_i \mid Z_S \subset gf(E_i) ; b_i > 0\}$. By changing the p_i slightly, we may assume that the minimum is achieved for exactly one index. Let us denote the corresponding divisor by E_0.

Put $W = \bigcup_{e_i - b_i < -1} gf(E_i)$.

Claim 1. $0 < c < 1$.

Proof of Claim 1. We prove $c > 0$. If $Z_S \subset gf(E_i)$, $b_i > 0$ and $e_i = -1$, then $f(E_i) \subset \Delta$. But this can not occur, because $g^{-1}(Z_S) \cap \Delta = \phi$ from $\mathrm{Bs}\,|mL| \cap \Delta = \phi$. **Q.E.D.**

Claim 2. Put $c' := \max\{(e_i + 1)/b_i \mid e_i + 1 < b_i\}$. Then $c \leqq c' < 1$ and c' is not affected by p_i's.

Claim 3. If W does not include $gf(E_i)$, then $cb_i - e_i + p_i < 1$ or $f(E_i) \subset \Delta$.

Proof of Claim 3. Here $e_i - b_i \geqq -1$. If $b_i \neq 0$, then $e_i - cb_i \geqq e_i - c'b_i > -1$. If $b_i = 0$ and $e_i > -1$, then $e_i - cb_i = e_i > -1$. If $b_i = 0$ and $e_i = -1$, then $f(E_i) \subset \Delta$. **Q.E.D.**

Claim 4. $gf(E_0) = Z_S$. If $cb_i - e_i + p_i \geqq 1$ and $i \neq 0$, then $gf(E_i)$ does not include Z_S.

Proof of Claim 4. If $cb_i - e_i + p_i \geqq 1$, then $gf(E_i) \subset W$ or $f(E_i) \subset \Delta$, by Claim 3. Because $Z_S \subset gf(E_0)$ and $g^{-1}(Z_S) \cap \Delta = \phi$, $gf(E_0) \subset W$. Here Z_S is an irreducible component of W. Thus $gf(E_0) = Z_S$. If $cb_i - e_i + p_i \geqq 1$ and $Z_S \subset gf(E_i)$, then $p_i < 1 + e_i$, because Δ does not include $f(E_i)$. So $b_i > 0$. Thus $c = (e_i + 1 - p_i)/b_i$ by the definition of c. Hence $i = 0$. **Q.E.D.**

Using Claims 1, 4, the same argument as in Kollár [5] implies the theorem. **Q. E. D.** for main theorem.

§3. Appendix.

Theorem 5. *Notation as in Theorem* 3. *Let Γ be a member of* **Strata**(Δ) *and* $d := \dim \Gamma$. *Then* $\mathrm{Bs}\,|mL|$ *does not include* Γ *for all* $m \geqq 2(d + a)$.

Remark. Blowing up with center Γ and using Theorems 3, 4, the theorem above follows.

Theorem 6 (Kollár-Matsuki [6, 4.12.1.2]. *Let* $f : Y \to X$ *be a birational* *morphism between non-singular projective varieties. Suppose that* $K_Y = f^*(K_X + \Delta) + \sum_{i=1}^{t} e_i E_i$ *and* Supp $\sum_{i=1}^{t} E_i$ *is with simple normal crossings. Then* $f(E_i) \in \mathbf{Strata}(\Delta)$ *for all* i *such that* $e_i = -1$.

Remark. Iitaka's Logarithmic Ramification formula [2] implies Theorem 6.

At first the author thought that these two theorems in this section are useful to get an estimate concerning Main theorem.

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