35. *An Example of Elliptic Curve over Q(T) with Rank $\geq$ 13*

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Abstract: We construct an elliptic curve over $Q(T)$ with rank ≥ 13.

In [1](resp. [2]), Mestre constructed elliptic curves over $Q(T)$ with rank ≥ 11 (resp. 12). In this paper, we construct an elliptic curve over $Q(T)$ with rank ≥ 13 using Mestre's method. As was explained in [1], for any 6-ple

$A = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \in Z^6$, we put $q_A(X) = \Pi_{i=1}^{6}(X - \alpha_i)$ and put $p_A(X) = q_A(X - T) * q_A(X + T) \in Q(T)[X]$. Then there are $g_A(X)$ and $r_A(X) \in Q(T)[X]$ with $\deg g_A = 6$, $\deg r_A \leq 5$ such that $p_A = g_A^2 - r_A$. Then the curve $Y^2 = r_A(X)$ contains 12 $Q(T)$-rational points $P_1, \ldots, P_{12}$ where $P_i = (T + \alpha_i, g_A(T + \alpha_i))$, $P_{i+6} = (-T + \alpha_i, g_A(-T + \alpha_i))$ $(1 \leq i \leq 6)$.

Let c_5 be the coefficient of X^5 of $r_A(X)$. By a suitable choice of A, we can assume that $c_5 = 0$. In the following, A will be always chosen so that $c_5 = 0$. Then $Y^2 = r_A(X)$ gives an elliptic curve over $Q(T)$ which will be denoted by $\mathscr{E}_A$.

Now, let $A = (148,116,104,57,25,0)$. (Then we have $c_5 = 0$.) In this case, the equation of the curve $\mathscr{E}_A$ and its $Q(T)$-rational points $P_1, \ldots, P_{12}$ are written as follows.

$$Y^2 = (9T^2 + 211950)X^4 + (-2700T^2 - 63901710)X^3 + (-18T^4 + 396150T^2 + 6706476489)X^2 + (2700T^4 - 29575350T^2 - 284435346600)X + 9T^6 - 159200T^4 + 891699592T^2 + 4156297690000,$$

$P_1 = [T + 148, 662T^2 + 66873T + 1868944]$

$P_2 = [T + 116, -554T^2 - 39687T - 191632]$

$P_3 = [T + 104, -526T^2 - 28497T + 163372]$

$P_4 = [T + 57, 508T^2 - 19332T - 368809]$

$P_5 = [T + 25, 580T^2 - 49116T + 566825]$

$P_6 = [T, -670T^2 + 69759T - 2038700]$

$P_7 = [-T + 148, -662T^2 + 66873T - 1868944]$

$P_8 = [-T + 116, 554T^2 - 39687T + 191632]$

$P_9 = [-T + 104, 526T^2 - 28497T - 163372]$

$P_{10} = [-T + 57, -508T^2 - 19332T + 368809]$

$P_{11} = [-T + 25, -580T^2 - 49116T - 566825]$

$P_{12} = [-T, 670T^2 + 69759T + 2038700]$.

By a direct calculation, we see that $\mathscr{E}_A$ contains another $Q(T)$-rational point

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$P_{13} = [(T + 703)/15, (-224T^3 - 844T^2 + 900484T + 2161725)/75]$.
(The existence of this point is important to break the record.)

Now we put $c_4 = 9T^2 + 211950$ (the coefficient of X^4 of r_A). The equation $S^2 = c_4(T)$ has a solution parametrized by T'

$$[T, S] = [(-T'^2 + 23550)/2T', 3(T'^2 + 23550)/2T'].$$

We consider the curve $\mathscr{E}'_A$ (defined over rational function field $Q(T')$) which is obtained from $\mathscr{E}_A$ by specialization $T \to (-T'^2 + 23550)/2T'$. Then, the 2-points P_∞ and $P_{\infty'}$ at infinity of $\mathscr{E}'_A$ become $Q(T')$-rational points in the same way as in [2].

Theorem. *$Q(T)$-rank of $\mathscr{E}'_A$ is* ≥ 13.

Proof. Now let E be the elliptic curve over Q obtained from $\mathscr{E}'_A$ by specialization $T' \to 1$ and let $p_i (i = 1,2,..13, \infty, \infty')$ be the rational-points of E obtained from P_i by the above specialization. Then in order to prove our theorem, we will show that $p_1, \ldots, p_{13}$ are independent points on E when group structure is given by p_∞ at origin.

It is known (cf. [3] p. 77) that the curve $Y^2 = a^2X^4 + bX^3 + cX^2 + dX + e (a, b, c, d, e \in Q)$ is Q-isomorphic to the Weierstrass model $Y^2 = X^3 + cX^2 + (bd - 4a^2e)X + (b^2e + a^2d^2 - 4a^2ce)$ by the map

$$\phi(X, Y) = (-2aY + 2a^2X^2 + bX, 4a^2XY + bY - 4a^3X^3 - 3abX^2 - 2acX - ad).$$

By this map, one of the points at infinity of the former curve goes to the unique point at infinity of Weierstrass model. Then we get a Weierstrass model of E. By using calculation system PARI, we see that the determinant of the matrix $(\langle \phi(p_i), \phi(p_j) \rangle)_{1 \leq i,j \leq 13}$ associated to the canonical height is 2910704763254221.2813489. Since this determinant is non-zero, we see that $p_1, \ldots, p_{13}$ are independent points, and we finish the proof.

References

[1] J. F. Mestre: Courbes elliptiques de rang ≥ 11 sur $Q(T)$. C. R. Acad. Sci. Paris, **313**, ser. 1, 139–142 (1991).

[2] ——: Courbes elliptiques de rang ≥ 12 sur $Q(T)$. ibid., **313**, ser. 1, 171–174 (1991).

[3] L. J. Mordell: Diophantine Equations. Academic Press (1968).