33. *Number Variance of the Zeros of the Epstein Zeta Functions*

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§1. Introduction. Let $Q(x, y) = ax^2 + bxy + cy^2$ be a positive definite quadratic form with discriminant $d = b^2 - 4ac$, where a, b and c are real numbers and $a > 0$. Then the Epstein zeta function $\zeta(s, Q)$ is defined by

$$\zeta(s, Q) = \frac{1}{2} \sum_{x,y}{}' Q(x, y)^{-s} \quad \text{for } \sigma = \Re(s) > 1,$$

where x, y runs over all integers excluding $(x, y) = (0,0)$ and $s = \sigma + it$ with real numbers σ and t. We put

$$k = \frac{\sqrt{|d|}}{2a}.$$

It has been the subject of many mathematicians to study the distribution of the zeros of $\zeta(s, Q)$ from the view point of the comparison with that of the Riemann zeta function $\zeta(s)$. $\zeta(s, Q)$ does not in general have Euler product expansion, while that of $\zeta(s)$ has been the key source for the proofs of various properties of the distribution of its zeros. Hence it is natural that $\zeta(s, Q)$ has in general the properties which the functions like $\zeta(s)$ never have. For example, $\zeta(s, Q)$ has in general a real zero between 0 and 1 (cf. Bateman-Grosswald [2]). In certain cases, $\zeta(s, Q)$ has even infinitely many zeros in $\Re(s) > 1$ (cf. Davenport and Heilbronn [4]). On the other hand, surprisingly enough, the Epstein zeta functions have in general also the properties which one has expected only to the functions like $\zeta(s)$. For example, $\zeta(s, Q)$ has infinitely many zeros on the critical line $\Re(s) = 1/2$ (cf. Kober [9]). More recently and more strongly, it has been shown under certain hypothesis by Bombieri and Hejhal [3] and Hejhal [7], that almost all the zeros of $\zeta(s, Q)$ with integral a, b and c lie on the critical line $\Re s = 1/2$. So we are left in the mist.

A remarkable result, bridging these two opposite directions, proved by Stark [13], is that "k-analogue" of the "Riemann Hypothesis" hold for the Epstein zeta functions. The purpose of the present article is to show that "k-analogue" of GUE law fails for the Epstein zeta functions. As we have shown in [5][6] (cf. also [1] and [11]), this should be distinguished completely from the other zeta functions like $\zeta(s)$.

We start with recalling Stark's "k-analogue" of the "Riemann Hypothesis". Stark [13] has shown that

for $k > K$, all the zeros of $\zeta(s, Q)$ in the region $-1 < \sigma < 2$, $-2k \leq t \leq 2k$ are simple zeros; with the exception of two real zeros between 0 *and* 1, *all* *are on the line* $\sigma = 1/2$ *and that for* $0 < T \leq 2k$,

$$N(T, Q) = \frac{T}{\pi}\log\left(\frac{kT}{\pi e}\right) + O(\log^{\frac{1}{3}}(T+3)(\log\log(T+3))^{\frac{1}{6}}),$$

where $N(T, Q)$ *denotes the number of the zeros of* $\zeta(s, Q)$ *in the region* $-1 < \sigma < 2$, $0 \leq t \leq T$.

As is seen in Stark's paper [13], we have for $0 < T \leq 2k$,

$$N(T, Q) = F_Q(T) + \Delta_Q(T),$$

where

$$F_Q(T) = \frac{1}{\pi}\arg\left(\frac{k}{\pi}\right)^{\frac{1}{2}+iT} + \frac{1}{\pi}\arg\Gamma\left(\frac{1}{2}+iT\right) + \frac{1}{\pi}\arg\zeta(1+i2T)$$

and

$$|\Delta_Q(T)| \leq C,$$

C being always some positive constant.

Since

$$\frac{1}{\pi}\arg\left(\frac{k}{\pi}\right)^{\frac{1}{2}+iT} + \frac{1}{\pi}\arg\Gamma\left(\frac{1}{2}+iT\right) = \frac{T}{\pi}\log\left(\frac{kT}{e\pi}\right) + O(1),$$

the number variance with which we are concerned is

$$\frac{1}{T}\int_T^{2T-2}\left(S_Q\left(t+\frac{\alpha\pi}{\log\frac{kT}{\pi}}\right) - S_Q(t)\right)^2 dt,$$

where we put

$$S_Q(t) = \frac{1}{\pi}\arg\zeta(1+i2t) + \Delta_Q(t).$$

If it obeys GUE law, then it must be that

$$\frac{1}{T}\int_T^{2T-2}\left(S_Q\left(t+\frac{\alpha\pi}{\log\frac{kT}{\pi}}\right) - S_Q(t)\right)^2 dt \sim C\log\alpha \text{ as } \alpha\to\infty.$$

Contrary to this, we can show the following theorem.

Theorem. *For* $k > K$ *and* $0 < T \leq k$, *there exists some positive constant* C *such that*

$$\frac{1}{T}\int_T^{2T-2}\left(S_Q\left(t+\frac{\alpha\pi}{\log\frac{kT}{\pi}}\right) - S_Q(t)\right)^2 dt \leq C$$

uniformly for positive $\alpha \leq \frac{1}{\pi}\log\frac{kT}{\pi}$.

Consequently, we see that as $k\to\infty$

$$\frac{1}{k}\int_k^{2k-2}\left(S_Q\left(t+\frac{\alpha\pi}{\log(k^2/\pi)}\right) - S_Q(t)\right)^2 dt \leq C$$

uniformly for positive $\alpha \leq (1/\pi)\log(k^2/\pi)$. Thus we see that "$k$-analogue" of GUE law fails for the Epstein zeta functions.

To prove the above theorem, we shall prove the following lemma which is more general than what we need.

Lemma. *For any* σ *in* $1/2 < \sigma \leq 1$, *there exists a positive constant* $\delta(\sigma)$ *which may depend on* σ *such that*

$$\int_T^{2T}(\arg(\zeta(\sigma+i(t+h))) - \arg(\zeta(\sigma+it)))^2\, dt$$

$$= T \sum_{n=2}^{\infty} \frac{\Lambda^2(n)}{n^{2\sigma} \log^2 n} (1 - \cos(h \log n)) + O(T^{1-\delta(\sigma)})$$

uniformly for $0 < h \ll T$, *where* $\Lambda(n)$ *is the von-Mangoldt function defined by*

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with a prime number } p \text{ and on integer } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that this for $\sigma = 1$ implies our Theorem stated above, since $\Delta_Q(t) = O(1)$.

§2. Proof of Lemma. Suppose that $2 \leq X = T^a \leq T^2$, a is a sufficiently small positive constant which may depend on σ. We put

$$\sigma_{X,t} = \frac{1}{2} + 2 \max_{\rho} \left(\beta - \frac{1}{2}, \frac{2}{\log X}\right),$$

ρ running here through all zeros $\beta + i\gamma$ of $\zeta(s)$ for which

$$|t - \gamma| \leq \frac{X^{3|\beta - \frac{1}{2}|}}{\log X}.$$

We put further

$$\Lambda_X(n) = \begin{cases} \Lambda(n) & \text{for } 1 \leq n \leq X \\ \Lambda(n) \dfrac{(\log (X^3/n))^2 - 2(\log (X^2/n))^2}{2(\log X)^2} & \text{for } X \leq n \leq X^2 \\ \Lambda(n) (\log (X^3/n))^2/2(\log X)^2 & \text{for } X^2 \leq n \leq X^3. \end{cases}$$

Under these notations, we shall use the following Selberg's explicit formula (cf. p. 239 of Selberg [12]) for $\sigma \geq \sigma_{X,t}$ and $t \geq \sqrt{X}$.

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n<X^3} \frac{\Lambda_X(n)}{n^s} + O\left(X^{\frac{1}{4}-\frac{\sigma}{2}} \left| \sum_{n<X^3} \frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}} \right|\right) + O(X^{\frac{1}{4}-\frac{\sigma}{2}} \log t).$$

Then we get for $t \geq T$ and for $\sigma \geq \sigma_{X,t}$,

$$\begin{aligned} \arg(\zeta(\sigma + it)) &= -\int_{\sigma}^{\infty} \Im \frac{\zeta'}{\zeta}(u + it)\, du \\ &= \Im \sum_{n<X^3} \frac{\Lambda_X(n)}{n^{\sigma+it} \log n} + O\left(\frac{X^{\frac{1}{4}-\frac{\sigma}{2}}}{\log X} \left(\left| \sum_{n<X^3} \frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}} \right| + \log t\right)\right) \\ &= M(t) + O(R(t)), \text{ say.} \end{aligned}$$

We put

$$f(\sigma, t) = \begin{cases} 1 & \text{if } \sigma \geq \sigma_{X,t} \\ 0 & \text{otherwise.} \end{cases}$$

Now for any $1/2 < \sigma \leq 1$,

$$\begin{aligned} &\int_T^{2T} (\arg(\zeta(\sigma + i(t + h))) - \arg(\zeta(\sigma + it)))^2\, dt \\ &= \int_T^{2T} f(\sigma, t + h) f(\sigma, t) (\arg(\zeta(\sigma + i(t + h))) - \arg(\zeta(\sigma + it)))^2\, dt \\ &+ \int_T^{2T} (1 - f(\sigma, t + h) f(\sigma, t)) (\arg(\zeta(\sigma + i(t + h))) - \arg(\zeta(\sigma + it)))^2\, dt \\ &= S_1 + S_2, \text{ say.} \end{aligned}$$

Since $\arg(\zeta(\sigma + it))$, $\arg(\zeta(\sigma + i(t + h))) \ll \log T$ for $\sigma \geq 1/2$, we get

$$S_2 \ll \log^2 T \int_T^{2T} (1 - f(\sigma, t))dt + \log^2 T \int_T^{2T} (1 - f(\sigma, t + h))dt$$
$$= \log^2 T(S_2' + S_2''), \text{ say.}$$
$$S_2' \leq \Big| \Big\{ T \leq t \leq 2T \text{ ; there exists } \beta + i\gamma \text{ in the region } \beta > \frac{1}{2} + \frac{1}{\log X}, \quad T - \frac{X^{\frac{3}{2}}}{\log X} \leq \gamma \leq 2T + \frac{X^{\frac{3}{2}}}{\log X} \text{ such that } \max_{\substack{|t-\gamma| \leq \frac{X^{3(\beta-\frac{1}{2})}}{\log X} \\ \beta > \frac{1}{2} + \frac{1}{\log X}}} \beta > \Big(\frac{1}{2} + \sigma\Big)\frac{1}{2} \Big\} \Big|$$
$$\leq 2 \sum_{\substack{T - \frac{X^{\frac{3}{2}}}{\log X} \leq \gamma \leq 2T + \frac{X^{\frac{3}{2}}}{\log X} \\ \beta > (\frac{1}{2}+\sigma)\frac{1}{2}}} \frac{X^{3(\beta - \frac{1}{2})}}{\log X} \ll \frac{X^{\frac{3}{2}}}{\log X} \Psi(\sigma, T),$$

where we put

$$\Psi(\sigma, T) = \min(T^{\frac{12}{5}(1-(\frac{1}{2}+\sigma)\frac{1}{2})}, T^{4(\frac{1}{2}+\sigma)\frac{1}{2}(1-(\frac{1}{2}+\sigma)\frac{1}{2})}) \log^C T.$$

We notice that we have used Theorem 1 in p. 128 and Theorem 1 in p. 131 of Karatsuba and Voronin [8]. (One might get a better estimate if one does not use a trivial estimate $X^{3(\beta-\frac{1}{2})} \ll X^{\frac{3}{2}}$.) In a similar manner we estimate S''_2 and get

$$S_2 \ll \frac{X^{3/2}}{\log X} \Psi(\sigma, T) \log^2 T.$$

To evaluate S_1, we use the above formula for $\arg(\zeta(\sigma + it))$ and $\arg(\zeta(\sigma + i(t + h)))$.We get first

$$S_1 = \int_T^{2T} f(\sigma, t + h) f(\sigma, h)(M(t + h) - M(t))^2 dt$$
$$+ O\Big(\sqrt{\int_T^{2T} (M(t + h) - M(t))^2 dt} \sqrt{\int_T^{CT} R(t)^2 dt}\Big) + O\Big(\int_T^{CT} R(t)^2 dt\Big)$$
$$= S_3 + O(\sqrt{S_4}\sqrt{S_5}) + O(S_5), \text{ say.}$$
$$S_3 = \int_T^{2T} (M(t + h) - M(t))^2 dt + \int_T^{2T} (f(\sigma, t + h) f(\sigma, t) - 1)(M(t + h) - M(t))^2 dt$$
$$= S_4 + S_6, \text{ say.}$$
$$S_6 \ll \sqrt{\int_T^{2T} (1 - f(\sigma, t))\, dt} \sqrt{\int_T^{2T} (M(t + h) - M(t))^4 dt} + \sqrt{\int_T^{2T} (1 - f(\sigma, t + h))\, dt} \sqrt{\int_T^{2T} (M(t + h) - M(t))^4 dt}$$
$$= \sqrt{S_2'}\sqrt{S_7} + \sqrt{S_2''}\sqrt{S_7}, \text{ say.}$$

So we are left to evaluate S_4, S_7 and S_5.

$$S_4 = \int_T^{2T} \Big(\frac{\eta(t) - \bar{\eta}(t)}{2i}\Big)^2 dt$$
$$= -\frac{1}{4}\int_T^{2T} \eta^2(t)\, dt - \frac{1}{4}\int_T^{2T} \bar{\eta}^2(t)\, dt + \frac{1}{2}\int_T^{2T} |\eta(t)|^2 dt$$
$$= S_8 + \bar{S}_8 + S_9, \text{ say,}$$

where we put

$$\eta(t) = \sum_{n<X^3} \frac{\Lambda_X(n)}{n^{\sigma+it}\log n}\left(\frac{1}{n^{ih}} - 1\right).$$

We get simply,

$$S_8 \ll \sum_{m,n<X^3} \frac{\Lambda_X(m)\Lambda_X(n)}{(mn)^{\sigma}\log m \log n \log(mn)} \ll \Phi(X, \sigma),$$

where we put

$$\Phi(X, \sigma) = \begin{cases} 1 & \text{if } \sigma = 1 \\ X^{6(1-\sigma)}/\log^3 X & \text{if } 1/2 < \sigma < 1. \end{cases}$$

By Montgomery and Vaughan [10], we get

$$\begin{aligned} S_9 &= \frac{1}{2}\sum_{n<X^3} (T + O(n))\frac{\Lambda_X^2(n)}{n^{2\sigma}\log^2 n}\left|\frac{1}{n^{ih}} - 1\right|^2 \\ &= T\sum_{n=2}^{\infty} \frac{\Lambda^2(n)}{n^{2\sigma}\log^2 n}(1 - \cos(h\log n)) \\ &\quad + O\left(T\sum_{n>X}\frac{\Lambda^2(n)}{n^{2\sigma}\log^2 n}\right) + O\left(\sum_{n<X^3}\frac{\Lambda^2(n)}{n^{2\sigma-1}\log^2 n}\right) \\ &= T\sum_{n=2}^{\infty} \frac{\Lambda^2(n)}{n^{2\sigma}\log^2 n}(1 - \cos(h\log n)) + O\left(\frac{T}{X^{2\sigma-1}\log X}\right) + O(\Phi_1(X, \sigma)), \end{aligned}$$

where we put

$$\Phi_1(X, \sigma) = \begin{cases} \log\log X & \text{if } \sigma = 1 \\ X^{6(1-\sigma)}/\log X & \text{if } 1/2 < \sigma < 1. \end{cases}$$

Using Montgomery and Vaughan [10] again, we get

$$\begin{aligned} S_7 &\ll \int_T^{2T}\left|\sum_{m,n<X^3}\frac{\Lambda_X(m)\Lambda_X(n)}{(mn)^{\sigma+it}\log m\log n}\left(\frac{1}{m^{ih}} - 1\right)\left(\frac{1}{n^{ih}} - 1\right)\right|^2 dt \\ &\ll \int_T^{2T}\left|\sum_{k<X^6}\frac{a(k)}{k^{\sigma+it}}\right|^2 dt \ll \sum_{k<X^6}(T + O(k))\frac{|a(k)|^2}{k^{2\sigma}} \\ &\ll T\sum_{k<X^6} d(k)^2 k^{-2\sigma} + \sum_{k<X^6} d(k)^2 k^{1-2\sigma} \ll T + X^{12(1-\sigma)}\log^3 X, \end{aligned}$$

where

$$a(k) = \sum_{\substack{mn=k \\ m,n<X^3}} \frac{\Lambda_X(m)\Lambda_X(n)}{\log m\log n}\left(\frac{1}{m^{ih}} - 1\right)\left(\frac{1}{n^{ih}} - 1\right) \ll d(k) \equiv \sum_{d|k} \cdot 1$$

and we have used the estimate

$$\sum_{k\le Y} d^2(k) \ll Y\log^3 Y.$$

Finally, we get, by using pp. 248-251 of Selberg [12],

$$\begin{aligned} S_5 &\ll \frac{X^{\frac{1}{2}-\sigma}}{\log^2 X}\left(\int_T^{CT}\left|\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}}\right|^2 dt + T\log^2 T\right) \\ &\ll \frac{X^{\frac{1}{2}-\sigma}}{\log^2 X}\left(\sqrt{T\log X}\left(\int_{\frac{1}{2}}^{\infty} X^{\frac{1}{2}-\sigma}\int_T^{CT}\left|\sum_{p<X^3}\frac{\Lambda_X(p)\log(Xp)}{p^{\sigma+it}\log^2 X}\right|^4 dt\, d\sigma\right)^{\frac{1}{2}} \right. \\ &\quad \left. + T\log^2 T\right) \ll TX^{(1/2)-\sigma}. \end{aligned}$$

Consequently, we get

$$\int_T^{2T} (\arg(\zeta(\sigma + i(t + h))) - \arg(\zeta(\sigma + it)))^2\, dt$$

$$= T\sum_{n=2}^{\infty}\frac{\Lambda^2(n)}{n^{2\sigma}\log^2 n}(1-\cos(h\log n)) + O\Big(T^{\frac{1}{2}+\frac{13a}{4}-\frac{7a\sigma}{2}}\frac{1}{\sqrt{\log T}}\Big)$$
$$+ O(T^{1-\frac{a}{2}(\sigma-\frac{1}{2})}) + O\Big(\frac{T^{6a(1-\sigma)}}{\log T}\Big) + O(T^{\frac{3a}{2}}\Psi(\sigma, T)\log T)$$
$$+ O\Big(T^{\frac{3a}{4}}\Big(T^{6a(1-\sigma)}\log T + \sqrt{\frac{T}{\log T}}\Big)\sqrt{\Psi(\sigma, T)}\Big).$$

Here we can choose an optimal a and get our Lemma as described in the introduction, although we shall not describe it explicitly.

§3. Concluding remarks. 3-1. It is clear that Stark's remainder term in $N(T, Q)$ can be replaced by $O(\log\log T)$.

3-2. More generally, we can evaluate the mean values

$$\int_T^{2T}(\arg(\zeta(\sigma+i(t+h))) - \arg(\zeta(\sigma+it)))^{2k}\,dt.$$

Here we notice only that we have the following asymptotic formula for an integer $k \geq 1$ and for any $1/2 < \sigma \leq 1$.

$$\int_T^{2T}(\arg(\zeta(\sigma+it)))^{2k}\,dt = TC(\sigma, k) + O(T^{1-\delta(\sigma,k)}),$$

where we put

$$C(\sigma, k) = \frac{(-1)^k}{2^{2k}}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^j\sum_{n_1,\ldots,n_j=2}^{\infty}\frac{\Lambda(n_1)\ldots\Lambda(n_j)}{(n_1\ldots n_j)^{2\sigma}\log n_1\ldots\log n_j}$$
$$\sum_{n_1\ldots n_j=m_1\ldots m_{2k-j}}\frac{\Lambda(m_1)\ldots\Lambda(m_{2k-j})}{\log m_1\ldots\log m_{2k-j}}$$

and $\delta(\sigma, k)$ is a positive constant which may depend on σ and k.

3-3. It is noticed by Professor Ramachandra that the remainder terms in the above mean value theorems for $\sigma = 1$ can be improved. For example, when $k = 1$ and $\sigma = 1$, the last remainder term can be replaced by $O(\log\log T)$.

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