30. On the Branching of Singularities in Complex Domains

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1. It is known that the singularities of the solution of the Cauchy problem in complex domains are generally contained in the union of the characteristic hypersurfaces K_i issued from the singular support T of the initial data (see [1], [5]-[7] and their references). But, usually, the singularities do not necessarily propagate onto all K_i. In fact, it is also known that there are, in general, solutions with singularities on and only on a given characteristic hypersurface (see [3], [4]).

In this note, we consider a special class of operators of second order with tangent characteristics, and show that the singularities of the solution always propagate onto both K_1 and K_2. This is a complex version of the branching of singularities.

2. We consider the partial differential equation

(1) $$Pu := \{D_t^2 + tD_xD_t + b(t, x)D_x + c(t, x)\}u(t, x) = 0$$

where $(t, x) \in \mathbf{C}^2$, $D_t = \partial/\partial t$, $D_x = \partial/\partial x$, $V_0 = \{(t, x) ; |t|, |x| < r_0\}$, $r_0 > 0$, $b, c \in H(V_0)$ and $H(V_0)$ denotes the set of all holomorphic functions in V_0. This equation has two characteristic curves $K_1 = \{x = 0\}$ and $K_2 = \{x - t^2/2 = 0\}$, which are mutually tangent at the origin.

Let $W_0 = \{x ; |x| < r_0\}$, $\dot{W}_0 = W_0 - \{0\}$, and $V = V_r = \{(t, x) ; |t|, |x| < r\}$, $r > 0$, and denote the universal covering space (revêtement universel) of $\dot{W}_0$ and of $V - K_1 \cup K_2$ by $\mathcal{R}(\dot{W}_0)$ and $\mathcal{R}(V - K_1 \cup K_2)$ respectively. Recall that $u(t, x) \in H[\mathcal{R}(V - K_1 \cup K_2)]$ implies u is holomorphic at a point $\zeta_0 = (0, x_0) \in (V - K_1 \cup K_2)$ and is analytically continued along any path issued from ζ_0 and traced in $V - K_1 \cup K_2$, and so does $v(x) \in H[\mathcal{R}(\dot{W}_0)]$.

On the Cauchy problem to obtain a solution of the equation (1) satisfying the initial condition

(2) $$D_t^i u(0, x) = v_i(x),\ i = 0,1,$$

the following theorem is known.

Theorem 0 (C. Wagschal [7]). *There exists $r > 0$ such that for any $v_i(x) \in H[\mathcal{R}(\dot{W}_0)]$, $i = 0,1$, the local solution of the Cauchy problem* (1)–(2) *around $\zeta_0 \in \dot{W}_0$ can be analytically continued to a function $u(t, x) \in H[\mathcal{R}(V_r - K_1 \cup K_2)]$.*

The question is thus if *the solution $u(t, x)$ is singular everywhere on $K_1 \cup K_2$ whenever at least one of $v_i(x)$ is singular at $x = 0$.* This question will be answered by employing

Definition. We say $u(t, x) \in H[\mathcal{R}(V-K_1 \cup K_2)]$ is regular at $\zeta_1 = (t_1, x_1) \in K_1 \cup K_2$, if u is analytically continued up to ζ_1 along any path γ : $\zeta = \zeta(s)\,(0 \leq s \leq 1)$ satisfying $\zeta(0) = \zeta_0$, $\zeta(1) = \zeta_1$ and $\zeta(s) \in (V - K_1 \cup K_2)$ for $0 \leq s < 1$. If u is not regular at ζ_1, we say it is singular there.

3. First, we have

Theorem 1. *Let $b(0,0) \notin \mathbf{Z}$. Then, if $u \in H[\mathcal{R}(V-K_1 \cup K_2)]$ is a solution of the equation* (1) *and regular at a point $\zeta_1 \in K_1 \cup K_2$, we have $u \in H(V)$. In other words, the solution $u(t, x) \in H[\mathcal{R}(V-K_1 \cup K_2)]$ to the Cauchy problem* (1)–(2) *is everywhere singular on $K_1 \cup K_2$ whenever at least one of $v_i(x) \in H[\mathcal{R}(\dot{W}_0)]$ is singular at $x=0$.*

We next consider the case

(3) $$b(0,0) \in \mathbf{Z} \text{ and } b_x(0,0) \neq 0.$$

Set

$$A_1(\lambda) = \lambda + b(0,0), \quad A_2(\lambda) = \lambda + 1 - b(0,0),$$

(4) $$B_1(\mu) = b_x(0,0)\mu + b_t^2(0,0) + (b(0,0) - 1/2)b_{tt}(0,0) + c(0,0),$$

$$B_2(\mu) = B_1(\mu + b(0,0) - 1/2).$$

Let λ_i denote the zero of $A_i(\lambda)$ and μ_i that of $B_i(\mu)$ respectively $(i = 1,2)$. Note, since $b(0,0) \in \mathbf{Z}$, one and only one of λ_i belongs to $\{0,1,2,\ldots\}$. Then we have

Theorem 2. *Suppose* (3) *and*

(5) $$\mu_i \notin \{0,1,2,\ldots\} \text{ for } i \text{ with } \lambda_i \in \{0,1,2,\ldots\},$$

then there exist $r > 0$ and a solution $u \in H[\mathcal{R}(V_r - K_i)] \setminus H(V_r)$ of the equation (1) *for the corresponding i. In a word, the consequence in Theorem* 1 *does not hold.*

We lastly consider the case

(6) $$b(0,0) \in \mathbf{Z} \text{ and } b_x(0, x) \equiv 0.$$

Since $B_i(\mu)$ is free of μ and i in this case, abbreviate it to B. Then we get

Theorem 3. *Suppose* (6) *and*

(7) $$B \neq 0,$$

then the same consequence in Theorem 1 *holds.*

Remark. For the Cauchy problem (1)–(2) with meromorphic initial data, J. Urabe [6] obtained an expression theorem of the solution assuming that $b_x(0, x) \equiv 0$. However, it does not seem so easy to derive the results stated above from his.

4. The complete proof will be given in a forthcoming paper. Here, let us briefly explain the way to prove the theorems. Firstly, one gets

Proposition 1. *Let $u \in H[\mathcal{R}(V-K_1 \cup K_2)]$, then the following* (a), (b) *and* (c) *are equivalent*:

(a) *u is regular at a point $\zeta_1 \in \dot{K}_2 := K_2 - \{(0,0)\}$.*

(b) *u is regular everywhere on $\dot{K}_2$.*

(c) $u \in H[\mathcal{R}(V-K_1)]$.

One may exchange K_1 with K_2 in the above statements.

Therefore Theorems 1 and 3 follow from the following proposition.

Proposition 2. *Under the assumption(s) in Theorem* 1 *or in Theorem* 3, *it holds for both $i=1$ and $i=2$ that*

$$u \in H[\mathscr{R}(V-K_i)],\ Pu=0 \Rightarrow u \in H(V).$$

This proposition for $i=1$, for example, is proved by considering the characteristic Cauchy problem

(8) $$Pu=0, \quad u|_{x=c}=u_0(t)$$

with a small parameter $c \in \mathbf{C}$. One can establish a Cauchy-Kowalewski type theorem with a uniform estimate of the existence domain of solutions with respect to c. (Under the assumption of Theorem 1, it is already done in [2].)

Theorem 2 is proved by constructing a singular solution. Namely, for $i=1$, for example, if $\mu_1 \notin \mathbf{Z}$, one can construct a solution of the equation (1) in the form

(9) $$u(t,x)=\sum_{j=0}^{\infty} u_j(t)x^{\mu_1+j}/\Gamma(\mu_1+j+1),\ u_0(t)\not\equiv 0.$$

But, if $\mu_1 \in \{-1,-2,-3,\cdots\}$, one must adopt the form

(10) $$u(t,x)=\sum_{j=\mu_1}^{-1} u_j(t)D_x^{-j-1}x^{-1}+\sum_{j=0}^{\infty}\{u_j(t)+v_j(t)\log x\}x^j/j!$$

with $u_{\mu_1}(t)\not\equiv 0$.

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