4. *Kähler Magnetic Fields on a Complex Projective Space*

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(Communicated by Heisuke HIRONAKA, M. J. A., Jan 12, 1994)

In this note we study trajectories of charged particles under the action of a Kähler magnetic field, a magnetic field corresponding to the Kähler form, on a complex projective space. We show that they are small circles on a totally geodesic embedded 2-dimensional sphere.

A magnetic field on a complete Riemannian manifold M is a closed 2-form B. Let $\Omega = \Omega_{B} : TM \to TM$ denote the skew symmetric operator on the tangent bundle satisfying $\boldsymbol{B}(X, Y) = \langle X, \Omega(Y)\rangle$. We call a curve γ on M a trajectory for this magnetic field if it is a solution of the equation $\nabla_{\dot\gamma}\dot\gamma = \Omega(\dot\gamma)$. Every trajectory γ has constant speed because $\frac{d}{dt}\|\dot\gamma(t)\|^2 = 2\langle\Omega(\dot\gamma(t)), \dot\gamma(t)\rangle = 0$. If γ is a trajectory of constant speed c for a magnetic field $\boldsymbol{B}$, the curve $\sigma(t) = \gamma(t/c)$ is a trajectory of unit speed for the magnetic field $c^{-1}\boldsymbol{B}$. We may therefore suppose trajectories are parametrized by their arc-length.

A magnenic field is called *uniform* if the associated skew symmetric operator is parallel $\nabla\Omega = 0$. Typical examples of uniform magnetic fields are scalar multiples of the volume form k-dvol on Riemann surfaces. On surfaces of constant curvature trajectories of such magnetic fields are well-known. On a sphere trajectories are small circles, on a Euclidean plane they are circles (in usual sense), and they are all closed. On a hyperbolic plane the feature is quite different. When the strength $|k|$ is greater than 1, trajectories are closed. But when it is not greater than 1 they are open (see [2] and also [5]).

We here give another example of uniform magnetic fields. Let (M, J) be a Kähler manifold and $\boldsymbol{B}_J$ denote the Kähler form; $\boldsymbol{B}_J(X, Y) = \langle X, JY\rangle$. Then the closed 2-form $\boldsymbol{B} = k\boldsymbol{B}_J$ with constant k is a uniform magnetic field. We shall call such field a *Kähler magnetic field*. It is quite natural to study trajectories for Kähler magnetic fields on manifolds of constant holomorphic sectional curvature. Trivially we can conclude that trajectories for a Kähler magnetic field are congruent on a manifold of constant holomorphic sectional curvature. That is, for given two trajectories γ and σ (of unit speed) for a Kähler magnetic field, we have a holomorphic isometry φ with $\sigma = \varphi\circ\gamma$.

In this note we show an explicit expression of trajectories for Kähler magnetic fields on a complex projective space. Let $\pi : S^{2n+1} \to \boldsymbol{C}P^n$ denote the Hopf fibration of a standard sphere onto a complex projective space. The tangent space of $\boldsymbol{C}P^n$ at $\pi(x)$ can be identified with the horizontal subspace of the tangent space of S^{2n+1} at x :

$$T_{\pi(x)}CP^n = \{[x, u] \mid u \in C^{n+1}, \langle x, u\rangle = 0\},$$

where $[x, u]$ denotes the orbit of (x, u) under the action $\lambda\cdot(x, u) = (\lambda x, \lambda u)$ of $U(1) = \{\lambda \in C \mid |\lambda| = 1\}$ on to the tangent bundle of the unit sphere.

Theorem. (1) *Every trajectory (of unit speed) for the Kähler magnetic field kB_J on a complex projective space $CP^n(4)$ of holomorphic sectional curvature* 4 *is a simple closed curve of period $2\pi/\sqrt{k^2+4}$.*

(2) *It lies on a totally geodesic embedded complex projective line.*

(3) *If $k \neq 0$, its horizontal lift on the sphere is a helix of order* 3 *with curvature $|k|$ and* 1.

(4) *The trajectory γ with $\gamma(0) = \pi(x)$ and $\dot\gamma(0) = [x, u] \in U_{\pi(x)}CP^n$ has the equation*

$$\gamma(t) = \pi((1+a^2)^{-1}(e^{ait} + a^2e^{bit})x + a(1+a^2)^{-1}(e^{bit} - e^{ait})Ju),$$

where $a = (k + \sqrt{k^2+4})/2$ and $b = (k - \sqrt{k^2+4})/2$.

Proof. Let $\tilde\nabla$ denote the connection of the standard sphere. For horizontal vector fields X and Y we have the following relation [4]:

$$\tilde\nabla_X Y = \nabla_X Y + \langle X, JY\rangle JN,$$

where N is the outward unit normal on $S^{2n+1} \subset C^{n+1}$. Using this relation we find that any horizontal lift $\tilde\gamma$ of a trajectory γ for $k\cdot B_J$ satisfies

$$\begin{cases} \tilde\nabla_{\dot{\tilde\gamma}}\dot{\tilde\gamma} = k\cdot J\dot{\tilde\gamma} \\ \tilde\nabla_{\dot{\tilde\gamma}}J\dot{\tilde\gamma} = -k\dot{\tilde\gamma} - JN \\ \tilde\nabla_{\dot{\tilde\gamma}}JN = J\dot{\tilde\gamma}, \end{cases}$$

which leads us to the third assertion. Regarding this curve on the sphere S^{2n+1} as a curve in C^{n+1} we see that it satisfies the equation $\ddot{\tilde\gamma}(t) = k\cdot J\dot{\tilde\gamma}(t) - \tilde\gamma(t)$. Under the initial condition $\tilde\gamma(0) = x$ and $\dot{\tilde\gamma}(0) = u$ we solve this linear ordinary differential equation and get that

$$\tilde\gamma(t) = (1+a^2)^{-1}(e^{ait} + a^2e^{bit})x + a(1+a^2)^{-1}(e^{bit} - e^{ait})Ju.$$

This expression guarantees that $\tilde\gamma$ lies on a 3 dimensional sphere, hence implies the second assertion. By this we can conclude that γ is a small circle of geodesic curvature k on a sphere of curvature 4, which leads us to the first assertion. (Paying an attention to the linearly independence of x, Ju, one can also check this assertion by a direct calculation.)

References

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