27. *On Hermitian Eisenstein Series*

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In [8], Shimura studied the analytic nature of the Eisenstein series for the Hermitian modular group. The main purpose of this note is to add some results in the theory of Hermitian Eisenstein series of low weights. The detailed proof will be given elsewhere.

1. Hermitian Eisenstein series. Let G_n be the group defined by

$$G_n = \{M \in SL_{2n}(\boldsymbol{C}) \mid {}^t\bar{M}J_nM = J_n\},\ J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

We write a typical element M of G_n in the form $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with matrices A, B, C, D of size n. Denote by P_n the subgroup of G_n consisting of the element M for which $C = 0$. Let $\boldsymbol{H}_n$ be the domain defined by

$$\boldsymbol{H}_n = \{Z \in M_n(\boldsymbol{C}) \mid I(Z) := (2i)^{-1}(Z - {}^t\bar{Z}) > 0\}.$$

This is called the *Hermitian upper half space of degree* n. The group G_n acts on $\boldsymbol{H}_n$ as

$$Z \mapsto M\langle Z\rangle = (AZ + B)(CZ + D)^{-1},\ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_n.$$

Let $\boldsymbol{K}$ be an imaginary quadratic number field of discriminant $d_{\boldsymbol{K}}$. We denote by $\mathcal{O}_{\boldsymbol{K}}$ the ring of integers in $\boldsymbol{K}$. The *Hermitian modular group of degree n over* $\boldsymbol{K}$ is defined by $\Gamma_n(\boldsymbol{K}) := G_n \cap M_{2n}(\mathcal{O}_{\boldsymbol{K}})$. We consider the following Eisenstein series:

$$E_k^{(n)}(Z, s) = \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : \Gamma_n(\boldsymbol{K})_0 \backslash \Gamma_n(\boldsymbol{K})} \det(CZ + D)^{-k} \mid \det(CZ + D)\mid^{-s},$$

where $(Z, s) \in \boldsymbol{H}_n \times \boldsymbol{C}$, $k \in 2\boldsymbol{Z}$ and $\Gamma_n(\boldsymbol{K})_0 = P_n \cap \Gamma_n(\boldsymbol{K})$. It is known that this series is absolutely convergent for $\mathrm{Re}(s) + k > 2n$ (cf. Braun [2]). Moreover, it can be continued as a meromorphic function in s to the whole complex plane (e.g. cf. Shimura [8]). We shall call this the *Hermitian Eisenstein series for* $\Gamma_n(\boldsymbol{K})$.

2. Functional equation. We introduce a functional equation of $E_k^{(n)}(Z, s)$, which plays an important role in the proof of our main result (Theorem 3). We denote by $\chi_{\boldsymbol{K}}$ the Kronecker symbol of $\boldsymbol{K}$. For $m \in \boldsymbol{Z}(m \geqq 0)$, we put

$$\rho(s\,;\chi_K^m) := \begin{cases} \pi^{-\frac{s}{2}}\Gamma(s/2)\zeta(s) & \text{if } m \text{ is even,} \\ \mid d_{\boldsymbol{K}}\mid^{\frac{s}{2}}\pi^{-\frac{s}{2}}\Gamma((s+1)/2)L(s\,;\chi_{\boldsymbol{K}}) & \text{if } m \text{ is odd,} \end{cases}$$

where $\Gamma(s)$ is the gamma function, $\zeta(s)$ is the Riemann zeta function, and $L(s\,;\chi_{\boldsymbol{K}})$ is the Dirichlet L-function with respect to $\chi_{\boldsymbol{K}}$. It is known that, for each m, the meromorphic function $\rho(s\,;\chi_{\boldsymbol{K}}^m)$ satisfies a functional equation $\rho(1-s\,;\chi_{\boldsymbol{K}}^m) = \rho(s\,;\chi_{\boldsymbol{K}}^m)$. Introduce a polynomial $\varepsilon_n(s) \in \boldsymbol{Z}[s]$ by $\varepsilon_n(s) = \Pi_{m=0}^{n-1}(s - m)$. Our first result is as follows:

Theorem 1. *Assume that the class number of $\boldsymbol{K}$ is one. Then the function*

$$\mathscr{E}_k^{(n)}(Z, s) = \prod_{m=1}^{n} \rho(s-m+1\,;\chi_{\boldsymbol{K}}^{m-1}) \prod_{j=1}^{k/2} \varepsilon_n((s/2)+j-1)\det I(Z)^{(s-k)/2} \cdot E_k^{(n)}(Z, s-k)$$

can be continued as a meromorphic function in s to the whole complex plane and satisfies a functional equation

$$\mathscr{E}_k^{(n)}(Z, 2n-s) = \mathscr{E}_k^{(n)}(Z, s).$$

Remark 1. This result is a Hermitian version of a result of Andrianov and Kalinin [1]. In the first place in our proof, we prove this in the special case $k=0$. This part is done by an argument analogous to Kim [5]. Afterward, we apply a differential operator studied by Shimura [9]. In the case $n=2$, this was proved in Nagaoka [6] by a direct method.

3. Hermitian Eisenstein series of low weight. As we stated before, the analytic property of $E_k^{(n)}(Z, s)$ was fully studied by Shimura in [8]. In fact, he proved the following results.

Theorem 2 (Shimura [8]). (1) *If $k \geqq n$, then $E_k^{(n)}(Z, s)$ is holomorphic in s at $s=0$.*
(2) *If $k > n+1$ or $k=n$, then $E_k^{(n)}(Z, 0)$ is holomorphic in Z and this has cyclotomic Fourier coefficients.*
(3) *Define a differential operator Δ_n on $\boldsymbol{H}_n$ by $\Delta_n = \det(\partial/\partial z_{ij})$ with respect to the variable matrix (z_{ij}) on $\boldsymbol{H}_n$. Then, there exist two holomorphic functions $p_1(Z)$ and $p_2(Z)$ on $\boldsymbol{H}_n$ such that $\Delta_n p_2 = 0$ and*

$$E_{n+1}^{(n)}(Z, 0) = \Delta_n\{p_1(Z) + p_2(Z)\log[\det(Z - {}^t\bar{Z})]\}.$$

Moreover, $\pi^n p_2(Z)$ is a $\Gamma_n(\boldsymbol{K})$-automorphic form of weight $n-1$ with cyclotomic Fourier coefficients and $\Delta_n p_1(Z)$ has a Fourier expansion with cyclotomic coefficients.
(4) *$E_{n-1}^{(n)}(Z, s)$ has at most a simple pole at $s=2$, and its residue is $\pi^{-n}\det(Z - {}^t\bar{Z})^{-1}$ times a $\Gamma_n(\boldsymbol{K})$-automorphic form f of weight $n-1$ with cyclotomic Fourier coefficients such that $\Delta_n f = 0$.*

Remark 2. Let f be a $\Gamma_n(\boldsymbol{K})$-automorphic form. The condition $\Delta_n f = 0$ means that f is a singular modular form (e.g. cf. Resnikoff [7]). The theory of singular modular forms was developed by Resnikoff [7], Freitag [3], and others. The theory of singular Hermitian modular forms was studied by Vasudevan [10] under the condition that the class number of $\boldsymbol{K}$ is one.

Our second result is as follows.

Theorem 3. *Assume that the class number of $\boldsymbol{K}$ is one. Then the Hermitian Eisenstein series $E_k^{(n)}(Z, s)$ has the following properties:*
(1) *$E_n^{(n)}(Z, s)$ is vanishing at $s=0$ if and only if n satisfies $n \equiv 2 \bmod 4$.*
(2) *$E_{n+1}^{(n)}(Z, 0)$ is holomorphic in Z if and only if n satisfies $n \equiv 3 \bmod 4$.*
(3) *$E_{n-1}^{(n)}(Z, s)$ is holomorphic at $s=2$ if and only if n satisfies $n \equiv 3 \bmod 4$.*

Remark 3. Our proof is based on analysis of the Fourier coefficients of $E_k^{(n)}(Z, s)$. An application of the theory of singular Hermitian modular forms shows the above results.

Example. Assume that $\boldsymbol{K} = \boldsymbol{Q}(i)$ (the Gaussian field). Iyanaga's matrix (cf. [4])

$$I=\begin{pmatrix} 2 & -i & -i & 1 \\ i & 2 & 1 & i \\ i & 1 & 2 & 1 \\ 1 & -i & 1 & 2 \end{pmatrix}$$

gives an example of positive definite, even integral, unimodular Hermitian matrix over $\boldsymbol{Z}[i]$. We consider a theta series on $\boldsymbol{H}_n$:

$$\Theta^{(n)}(Z\,;I)=\sum_{X\in M_{4\times n}(Z[i])}\exp[\pi i\,\mathrm{tr}({}^t\bar{X}IXZ)],\ Z\in\boldsymbol{H}_n.$$

Then we have the following results:

(1) $E_2^{(2)}(Z,\,0)=0$ and $E_4^{(4)}(Z,\,0)=2\Theta^{(4)}(Z\,;I)$.

(2) $E_4^{(3)}(Z,\,0)=\Theta^{(3)}(Z\,;I)$ and

$E_6^{(5)}(Z,\,0)=\Delta_5\{p_1(Z)+\pi^{-5}\Theta^{(5)}(Z\,;I)\log[\det(I(Z)]\}$.

(3) $\mathrm{Res}_{s=2}E_2^{(3)}(Z,\,s)=0$ and

$$\begin{aligned}\mathrm{Res}_{s=2}E_4^{(5)}(Z,\,s)&=\pi^6|d_K|^{\frac{5}{2}}\Gamma(5)^{-1}\zeta(6)^{-1}L\Big(5\,;\Big(\frac{-4}{*}\Big)\Big)^{-1}\det I(Z)^{-1}\Theta^{(5)}(Z\,;I)\\&=12096i\cdot\pi^{-5}\det(Z-{}^t\bar{Z})^{-1}\Theta^{(5)}(Z\,;I).\end{aligned}$$

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