1. *On Real Submanifolds of Kähler Manifolds Foliated by Complex Submanifolds*

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This work is motivated by the study of topological properties of Levi-flat hypersurfaces of complex manifolds (see e.g. [1] and [2]). Let X be a Kähler manifold and M a compact orientable real submanifold of X. We suppose that M admits a foliation $\mathcal{F}$ whose leaves are complex submanifolds of X. In this setting, we are interested in investigating the relation between the topology of $\mathcal{F}$ and the complex structure of X. We first observe that the exactness of the Kähler form implies the intricacies of $\mathcal{F}$ (Theorem 1 and Remark), and then apply it to the nonexistence problem of Levi-flat hypersurfaces in CP^2 (Corollary 2).

Theorem 1. *Let X, M and $\mathcal{F}$ be as above, and let $p = \dim_C \mathcal{F}$. If the p-th power of the Kähler form of X is exact when restricted to M, then $\mathcal{F}$ has no nontrivial foliation cycles.*

See e.g. [5] for the definition of foliation cycles.

Proof. Suppose $\mathcal{F}$ has a nontrivial foliation cycle C. Since each leaf of $\mathcal{F}$ is a complex submanifold of X, the p-th power of the Kähler form ω of X restricts to a volume form on each leaf. Then, by the local integration formula of foliation cycles ([5, Theorem I.12]) we see that the nontriviality of C implies the nonvanishing of the homological pairing $\langle C, \omega^p \mid M\rangle$. Thus, $\omega^p \mid M$ cannot be exact and the proof is complete.

Remark. It is known ([3], [5]) that if a foliation $\mathcal{F}$ on a compact manifold M admits no nontrivial foliation cycles, $\mathcal{F}$ must satisfy the following properties:

(1) Every leaf of $\mathcal{F}$ has exponential growth . In particular, $\mathcal{F}$ has no compact leaves.

(2) If codim $\mathcal{F} = 1$, some leaf of $\mathcal{F}$ has nontrivial holonomy.

(3) If codim $\mathcal{F} = 1$ and $\dim M = 3$, $\pi_1(M)$ has exponential growth.

Corollary 1. *Let M be a compact smooth real hypersurface of $CP^n (n \geq 2)$. Suppose that the Levi form on M has constant rank $k (0 \leq k \leq [n/2] - 1]$ and hence defines a codimension $(2k+1)$ foliation $\mathcal{F}$ on M by complex submanifolds of CP^n([4]). Then, $\mathcal{F}$ has no nontrivial foliation cycles.*

Proof. We denote by $i : M \rightarrow CP^n$ the inclusion map and by ω the standard Kähler form of CP^n. By Theorem 1, it suffices to show that $i^*\omega^{n-k-1}$ is exact for $0 \leq k \leq [n/2] - 1$. Suppose the contrary. One can find a $2(n-k-1)$-cycle α of M such that $\langle i^*\omega^{n-k-1}, \alpha\rangle \neq 0$. Then, $\langle \omega^{n-k-1}, i_* \alpha\rangle \neq 0$. Hence, $i_*[\alpha]$ is nontrivial in $H_{2(n-k-1)}(CP^n ; \mathbf{Z})$. From this it follows that the

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self-intersection of $i_*[\alpha]$ does not vanish. On the other hand, since the normal bundle of a real hypersurface in CP^n is always trivial, by deforming $i_*\alpha \subset M$ along a normal vector field to M one can easily remove the self-intersection of $i_*[\alpha]$ geometrically. This is a contradiction, which completes the proof.

It has been conjectured (by folklore) that CP^2 admits no compact smooth Levi-flat hypersurfaces. Our argument can show the truth of the conjecture in the case when $\pi_1(M)$ is not big:

Corollary 2. *Let M be a compact orientable 3-manifold such that $\pi_1(M)$ has nonexponential growth. Then M cannot be embedded in CP^2 as a smooth Levi-flat hypersurface.*

Proof. This follows from Corollary 1 ($n = 2$, $k = 0$) and Remark (3).

Corollary 3. *Let X be a Kähler manifold, M a compact orientable real submanifold of X. Suppose that M admits a foliation $\mathcal{F}$ by p-dimensional comlex submanifolds of X. If there is an integer r, $1 \leq r \leq p$, such that the $2r$-th Betti number of M vanishes, then $\mathcal{F}$ has no nontrivial foliation cycles.*

Proof. We denote by $i : M \to X$ the inclusion map and by ω the Kähler form of X. Suppose that the $2r$-th Betti number of M is zero for some $1 \leq r \leq p$. Then, $i^*\omega^r$ is exact, and hence so is $i^*\omega^p$. Now the conclusion follows from Theorem 1.

Evidently, statements similar to Theorem 1 hold also in symplectic geometry, whose proofs are the same as that of Theorem 1:

Theorem 2. *Let X be a symplectic manifold, M a compact orientable submanifold of X. Suppose that M admits a foliation $\mathcal{F}$ by $2p$-dimensional symplectic submanifolds of X. If the p-th power of the symplectic form of X is exact when restricted to M, then $\mathcal{F}$ has no nontrivial foliation cycles.*

Theorem 3. *Let X be a contact manifold endowed with a contact form η, M a compact orientable submanifold of X. Suppose that M admits a foliation $\mathcal{F}$ by symplectic submanifolds of X (i.e $d\eta$ restricted to each leaf defines a symplectic structure). Then $\mathcal{F}$ has no nontrivial foliation cycles.*

References

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