18. *Periodic Solutions of the 2-dimensional Heat Convection Equations*

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(Communicated by Kiyosi ITÔ, M. J. A., March 12, 1993)

§1. Introduction. We consider the heat convection equation in a time-dependent domain $\Omega(t) \subset R^2$. We assume that the domain $\Omega(t)$ varies periodically in t with period T. In the 3-dimensional case, we proved the existence and uniqueness of the periodic strong solution in [7] and showed the stability of it in [8] when the data were small in a suitable sense. In this paper, under somewhat released conditions than 3-dimensional case, we have studied the existence, uniqueness and the stability of the periodic strong solution. Recently, Morimoto [5] has got the periodic weak solution and Inoue-Ôtani [3] obtained the periodic strong one under their various situations.

§2. Assumptions and formulation. Let $\Omega(t)$ be a time-dependent bounded space domain in R^2 with the boundary $\partial\Omega(t) = \Gamma_0 \cup \Gamma(t)$, where Γ_0 is the inner boundary and $\Gamma(t)$ is the outer one. We denote by K the compact set which is bounded by Γ_0. We suppose that $\Omega(t)$ is included in a fixed open ball B_1 with radius d such that $\Omega(t) \subset B_1$. We make the following assumptions:

(A0) Γ_0 and $\Gamma(t)$ do not intersect each other.

(A1) For each fixed $t > 0$, $\Gamma(t)$ and Γ_0 are both simple closed curves and they are of class C^3.

(A2) $\Gamma(t) \times \{t\}\,(0 < t < T)$ changes smoothly (say, of class C^4) with respect to t.

(A3) $g(x)$ is a bounded and continuous vector function in $R^2 \setminus \operatorname{int} K$.

(A4) $\beta(x, t)$ is defined on $\partial\Omega(t)$ and it can be extended to a vector function $b = b(x, t)$ of the form $b = \operatorname{rot} c$, where $c(x, t)$ is defined in $B \times [0, \infty)$, of class C^3 and periodic in t with period T. Moreover, it satisfies the following condition

$$\int_{\Gamma_i} \beta\cdot n \, ds = 0, \; i = 0, 1,$$

where Γ_1 means $\Gamma(t)$ and n is the outer normal vector to $\partial\Omega(t)$.

(A5) The domain $\Omega(t)$, the boundary $\Gamma(t)$ and the function $\beta(x, t)$ vary periodically in t with period $T > 0$. i.e., $\Omega(t+T) = \Omega(t)$, $\Gamma(t+T) = \Gamma(t)$ and $\beta(\cdot, t+T) = \beta(\cdot, t)$ for each $t > 0$.

Now, let $u = u(x, t)$, $\theta = \theta(x, t)$ and $p = p(x, t)$ be the velocity of the viscous fluid, the temperature and pressure, respectively. Furthermore, let $\nu, \kappa, \alpha, \rho$ be physical constants and $g = g(x)$ be the gravitational vector. Then we consider the heat convection equation (HC) of Boussinesq approximation:

$$\text{(HC)}\begin{cases} u_t + (u\cdot\nabla)u = -\nabla p/\rho + \{1-\alpha(\theta - T_0)\}g + \nu\Delta u, \\ \operatorname{div} u = o, \\ \theta_t + (u\cdot\nabla)\theta = \kappa\Delta\theta, \end{cases}$$

in $\hat{\Omega} = \bigcup_{o<t<T} \Omega(t) \times \{t\}$ with the boundary condition

(1) $u|_{\partial\Omega(t)} = \beta(x, t)$, $\theta|_{\Gamma_0} = T_0 > 0$, $\theta|_{\Gamma(t)} = 0$ for any $t > 0$,

and with the periodicity condition

(2) $u(\cdot, t+T) = u(\cdot, t)$, $\theta(\cdot, t+T) = \theta(\cdot, t)$, in $\Omega(t+T) = \Omega(t)$.

On the function β, the next result holds (Lemma 2.7 in [4]).

Lemma 1. *For any $\eta > 0$, we can reconstruct a periodic (with period T) function $b \in W_2^2(B)$ satisfying $b = \beta$ on $\partial\Omega(t)$,* div $b = 0$ *and* $|((u\cdot\nabla)b, u)| \leqq \eta \|\nabla u\|^2$ *for any $u \in H_\sigma^1(\Omega(t))$.*

We choose a smooth periodic (with period T) function $\bar{\theta}$ on $\Omega(t)$ with the same boundary values on $\partial\Omega(t)$ as θ. Then, making a suitable change of variables (see, (3.7), (3.8) in [7]) and using the same letters after changing of variables, we get the heat convection equation of the following type:

$$(3)\quad\begin{cases} u_t + (u\cdot\nabla)u = -\nabla p - R\theta + \Delta u - (u\cdot\nabla)b - (b\cdot\nabla)u \\ \qquad\qquad -b_t - (b\cdot\nabla)b + \Delta b + d^3 g/\nu^2 - R(\bar{\theta} - \kappa/\nu) \\ \operatorname{div} u = o, \\ \theta_t + (u\cdot\nabla)\theta = (\kappa/\nu)\Delta\theta - (u\cdot\nabla)\bar{\theta} - (b\cdot\nabla)\theta - (b\cdot\nabla)\bar{\theta}, \end{cases}$$

where $R = \alpha g T_0 d^3/\kappa\nu$.

The boundary condition (1) is replaced by the following:

(4) $u|_{\partial\Omega(t)} = 0$, $\theta|_{\partial\Omega(t)} = 0$ for any $t > 0$.

On the other hand, the periodicity condition is still the same one as (2).

We prepare some convex functions to define the strong solution of (HC). (We use symbols $W_p^k(\Omega)$, $\mathring{W}_p^k(\Omega)$, $H_\sigma(\Omega)$ and $H_\sigma^1(\Omega)$, as usual.) Put $U = {}^t(u, \theta)$ and notice $H_\sigma(B) \times L^2(B) = (H_\sigma(B) \times O) + (O \times L^2(B))$ (direct sum) where $B = B_1 \backslash K$. Then, we introduce a proper lower semicontinuous convex (p. l. s. c.) function as follows:

$$(5)\quad \varphi_B(U) = \begin{cases} \dfrac{1}{2}\displaystyle\int_B (|\nabla u|^2 + \frac{\kappa}{\nu}|\nabla\theta|^2)dx \text{ if } U \in H_\sigma^1(B) \times \mathring{W}_2^1(B), \\ +\infty \text{ if } U \in (H_\sigma(B) \times L^2(B)) \backslash (H_\sigma^1(B) \times \mathring{W}_2^1(B)). \end{cases}$$

Moreover, we consider a closed convex set $K(t)$ in $H_\sigma(B) \times L^2(B)$:

(6) $K(t) = \{U \in H_\sigma(B) \times L^2(B)\,;\, U = 0 \text{ a.e. in } B\backslash\Omega(t)\}$

for any $t \geq 0$ and define its indicator function $I_{K(t)}$, namely, $I_{k(t)} = 0$ if $U \in K(t)$ and $I_{K(t)} = +\infty$ if $U \in (H_\sigma(B) \times L^2(B)) \backslash K(t)$. Then we define a p.l.s.c. function φ^t by

(7) $\varphi^t(U) = \varphi_B(U) + I_{K(t)}(U)$ for any $t \geq 0$.

Let $\partial\varphi^t$ be the subdifferential operator of φ^t, then it holds that

(i) $D(\partial\varphi^t) = \{U \in H_\sigma(B) \times L^2(B)\,;\, U|_{\Omega(t)} \in (W_2^2(\Omega(t)) \cap H_\sigma^1(\Omega(t))) \times (W_2^2(\Omega(t)) \cap \mathring{W}_2^1(\Omega(t))),\ U|_{B\backslash\Omega_{(t)}} = 0\}$,

(ii) $\partial\varphi^t(U) = \{f \in H_\sigma(B) \times L^2(B)\,;\, P(\Omega(t))f|_{\Omega(t)} = A(\Omega(t))U|_{\Omega(t)}\}$

where $A(\Omega(t)) = {}^t(-P_\sigma(\Omega(t))\Delta, -(\kappa/\nu)\Delta)$, $P(\Omega(t)) = {}^t(P_\sigma(\Omega(t)), 1_{\Omega(t)})$

and $P_\sigma(\Omega(t))$ stands for the orthogonal projection from $L^2(\Omega(t))$ to $H_\sigma(\Omega(t))$. Then, the equations (3) and (4) can be reduced to the following abstract heat convection equation (AHC) in $H_\sigma(B) \times L^2(B)$:

(AHC) $\dfrac{dU}{dt} + \partial\varphi^t(U(t)) + F(t)U(t) + M(t)U(t) \ni P(B)\tilde{f}(t),\ t \geq 0$

where $U = {}^t(u, \theta)$, $F(t)U(t) = {}^t(P_\sigma(B)(u \cdot \nabla)u, (u \cdot \nabla)\theta)$, $M(t)U(t) = {}^t(P_\sigma(B)((u \cdot \nabla)b + (b \cdot \nabla)u + R\theta), (u \cdot \nabla)\tilde{\bar{\theta}} + (b \cdot \nabla)\theta)$, $\tilde{f} = {}^t(\tilde{f}_1, \tilde{f}_2) = (-b_t - (b \cdot \nabla)b + \Delta b + d^3 g/\nu^2 - R(\tilde{\bar{\theta}} - \kappa/\nu), -(b \cdot \nabla)\tilde{\bar{\theta}})$; $\tilde{f}$ and $\tilde{\bar{\theta}}$ denote extensions of f and $\bar{\theta}$ to B with zero outside $\Omega(t)$, respectively.

Under these preparations, we define the strong solution of (AHC) as follows.

Definition 1. Let $U: [0, S] \to H_\sigma(B) \times L^2(B)$, $S \in (0, \infty)$. Then U is called a strong solution of (AHC) on $[0, S]$ if it satisfies the following properties (i) and (ii).

(i) $U \in C([0, S]; H_\sigma(B) \times L^2(B))$ and dU/dt exists for a.e. $t \in (0, S]$.

(ii) $U(t) \in D(\partial\varphi^t)$ for a.e. $t \in [0, S]$ and there exists a function $G: [0, S] \to H_\sigma(B) \times L^2(B)$ satisfying $G(t) \in \partial\varphi^t(U(t))$ and $(dU/dt) + G(t) + F(t)U(t) + M(t)U(t) = P(B)\tilde{f}(t)$ for a.e. $t \in [0, S]$.

Definition 2. A strong solution of (AHC) is called a periodic strong solution (resp. a strong solution of the initial value problem) if it satisfies the condition (8) (resp. (9)) stated below:

(8) $U(t+T) = U(t)$ for $t \in [0, \infty)$ in $H_\sigma(B) \times L^2(B)$,

(9) $U(0) = {}^t(\tilde{a}, \tilde{h})$ in $H_\sigma(B) \times L^2(B)$,

where a and h are prescribed initial data in $H_\sigma(\Omega(0)) \times L^2(\Omega(0))$.

§3. Results. First, we notice the Poincaré inequality.

Lemma 2. *There exists a positive constant* c_1 *such that*

(10) $\varphi^t(U) \geq c_1 \| U \|^2_{L^2(B)}$

holds for every $t \in [0, T]$ *and* $U \in H^1_\sigma(B) \times \mathring{W}^1_2(B)$.

Now we make the following assumption on b.

(A6) $b \in L^\infty(0, T; W^2_2(B))$ and $b_t \in L^\infty(0, T: L^2(B))$.

Then, our results can be stated as follows.

Theorem 1. (i) *There exists a positive number* T_0 *such that if* $T \geqq T_0$ *and if* (A0)~(A6) *are satisfied, then* (AHC) *has a periodic strong solution* U_Π *with period* T.

(ii) *Moreover, if* $\| b \|_{L^\infty(0,T;W^2_2(B))}$, $\| b_t \|_{L^\infty(0,T;L^2(B))}$ *and* $\| \nabla \tilde{\bar{\theta}} \|_{L^\infty(0,T;L^2(B))}$ *are sufficiently small and* ν *is large enough, then the periodic strong solution is unique.*

Remark 1. T_0 does not depend on the magnitude of the function b.

Theorem 2. (i) *Let* U_Π *be any periodic strong solution in* (i) *of Theorem* 1. *Then for any* $U_0 \in H_\sigma(\Omega(0)) \times L^2(\Omega(0))$, *there exists a unique strong solution* U *on* $[0, \infty)$ *with* $U(0) = U_\Pi(0) + U_0$.

(ii) *Moreover, if* b *and* $\tilde{\bar{\theta}}$ *are small in the same sense as in* (ii) *of Theorem* 1 *and* ν *is large enough, then* $\| U(t) - U_\Pi(t) \|_{L^2(\Omega(t)) \times L^2\Omega(t))} \to 0$ *as* $t \to \infty$.

§4. Proof of the theorems. We prepare some lemmas to prove the theorems.

Lemma 3. *Let $U = {}^t(u, \theta)$ be a strong solution of (AHC). Then we have the following estimate on θ.*
(11) $\| \theta(t) \|_{L^2(B)} \leq | B |^{1/2}\kappa/\nu + \| \theta(0) \| \exp(-2\kappa t/\nu)$,
where $| B |$ is the volume of the domain B.

The above lemma is a version of Lemma 2.1 of [1]. (See also [9].)

Lemma 4. (i) *Let $U = {}^t(u, \theta)$ be a strong solution of (AHC). Then, for any $\delta \in (0, S)$, there are positive constants $a_i(\delta)\,(i = 1,2,3)$, independent of S, depending on b and $\bar{\theta}$, such that*
(12) $\varphi^t(U(t)) \leq (a_2(\delta)\delta^{-1} + a_3(\delta))\exp(a_1(\delta))$
holds for every $t \in [\delta, S]$.
(ii) *Furthermore, if U is a periodic strong solution with period T, then the estimate* (12) *is valid for all $t \in [0, T]$.*

Proof of Lemma 4. The claim (i) can be proved by applying Lemma 9 of [9] essentially. We will show (ii). We notice that if U is periodic, then $\varphi^{t+T}(U(t+T)) = \varphi^t(U(t))$ for any $t \in [0, \infty)$. On the other hand, since b and $\bar{\theta}$ are periodic we find from (A6) that $b \in L^\infty(0, \infty\,; W_2^2(B))$, $b_t \in L^\infty(0, \infty\,; L^2(B))$ and $\tilde{\bar{\theta}} \in L^\infty(0, \infty\,; C^1(B))$, hence (12) holds for $t \in [\delta, \infty)$. Therefore, (12) is affirmative for all $t \in [0, T]$.

Remark 2. In Lemma 4, we can show that $a_i(\delta)$ are uniformly bounded in any small positive δ and that $a_i(\delta)\,(i = 1,2,3)$ are small if $\| b \|_{L^\infty(0,T;W_2^2(B))}$, $\| b_t \|_{L^\infty(0,T;L^2(B))}$ and $\| \nabla \tilde{\bar{\theta}} \|_{L^\infty(0,T;L^2(B))}$ are sufficiently small and ν is large enough. We omit verification. (See [1].)

Proposition 1. *Let $U_0 = {}^t(a, h) \in H \equiv H_\sigma(\Omega(0) \times L^2(\Omega(0))$. Then, (AHC) has a unique strong solution $U = {}^t(u, \theta)$ on $[0, S]$ satisfying $U(0) = U_0$, where $S > 0$.*

Outline of the proof of Proposition 1. For a given $U_0 \in H$, there exists a sequence $\{U_{0,n}\} \subset H_\sigma^1(\Omega(0)) \times \mathring{W}_2^1(\Omega(0))$ such that $\| U_{0,n} - U_0 \|_H \to 0$ as $n \to \infty$. Then, we have strong solutions U_n of (AHC) with $U_n(0) = U_{0,n}$ (see [8]). On the other hand, by Gronwall's inequality, we obtain

$$\| U_n(t) - U_m(t) \|_H \leq C \| U_{0,n} - U_{0,m} \|_H \text{ for any } t \in [0, S],$$

where $C > 0$ is a constant independent of n, m, t, which implies that there exists $U \in C([0, S]\,; H)$ satisfying $\| U_n(t) - U(t) \|_H \to 0$ as $n \to \infty$ uniformly on $[0, S]$. We will show U is a solution of (AHC). Indeed, if we take an arbitrary $\delta \in (0, S)$, then, using Lemma 4 together with the boundedness of $\{U_{0,n}\}$ and the lower semicontinuity of φ^t, we get
(13) $\varphi^t(U(t)) \leq \liminf_{n\to\infty} \varphi^t(U_n(t)) \leq (a_2(\delta)/\delta + a_3(\delta))e^{a_1(\delta)}$
for all n and $t \in [\delta, S]$. Recall Remark 2, then $a_i(\delta)$ are uniformly bounded in $\delta \in (0, S)$ and this implies that $U(t) \in D(\varphi^t)$ for $t \in (0, S]$. The remaining part of the proof is easy, so we omit it.

Proof of Theorem 1. First we prove (i). Let ε be an appropriate positive number such that $\varepsilon < \min(4\kappa/c_1\nu, 4)$. We put
(14) $A_1 = 4((4-\varepsilon)\varepsilon c_1^2)^{-1}\{\| \tilde{f} \|^2_{L^\infty(0,T:L^2(B)\times L^2(B))} + 2(| R |^2 + \| \nabla \tilde{\bar{\theta}} \|^2_{L^\infty(\hat{B})}) | B | \kappa^2/\nu^2\}$,
(15) $A_2 = 8((4-\varepsilon)c_1)^{-1}(| R |^2 + \| \nabla \tilde{\bar{\theta}} \|^2_{L^\infty(\hat{B})})(4\kappa/\nu - \varepsilon c_1)^{-1}$

where $\hat{B} = B \times [0, T]$.
Multiplying the both sides of (AHC) by $U(T)$ and integrating on B, then we get

$$\frac{1}{2}\frac{d}{dt}\| U(t) \|^2 + 2\varphi^t(U(t))$$
$$\leq |((u\cdot \nabla)b, u)| + |R|\cdot|(\theta, u)| + |((u\cdot\nabla)\tilde{\tilde{\theta}}, \theta)| + |(\tilde{f}, U)|.$$

We notice for some $c_1' > 0$, $c_1'\|\nabla u(t)\|^2 \leqq \varphi^t(U(t))$ holds. We use Lemma 1 with $\eta = (4-\varepsilon)c_1'/8$, then we have

$$\frac{1}{2}\frac{d}{dt}\| U(t) \|^2 + 2\varphi^t(U(t)) \leq \frac{4-\varepsilon}{8}(c_1'\|\nabla u(t)\|^2 + 3c_1\|u(t)\|^2)$$
$$+ \frac{2}{(4-\varepsilon)c_1}\{(|R|^2 + \|\nabla \tilde{\tilde{\theta}}\|^2_{L^\infty(\hat{B})})\|\theta(t)\|^2 + \|\tilde{f}(t)\|^2\},$$

from which we get

$$\frac{d}{dt}\| U(t) \|^2 + \varepsilon c_1\| U(t) \|^2 \leq \frac{4}{(4-\varepsilon)c_1}\{(|R|^2 + \|\nabla \tilde{\tilde{\theta}}\|^2_{L^\infty(\hat{B})})\|\theta(t)\|^2$$
$$+ \|\tilde{f}(t)\|^2\}.$$

Hence, using the elementary calculation, we have

$$(16)\quad \| U(t) \|^2 \leq e^{-\varepsilon c_1 t}\| U(0) \|^2$$
$$+ e^{-\varepsilon c_1 t}\int_0^t e^{\varepsilon c_1 s}\frac{4}{(4-\varepsilon)c_1}\{(|R|^2 + \|\nabla \tilde{\tilde{\theta}}\|^2_{L^\infty(\hat{B})})\|\theta(s)\|^2 + \|\tilde{f}(s)\|^2\}ds.$$

Employing Lemma 3, we obtain

$$(17)\quad \| U(t) \|^2 \leq e^{-\varepsilon c_1 t}\| U(0) \|^2 + \frac{4}{(4-\varepsilon)\varepsilon c_1^2}\{\|\tilde{f}\|^2_{L^\infty(0,\infty;L^2(B)\times L^2(B))}$$
$$+ 2(|R|^2 + \|\nabla \tilde{\tilde{\theta}}\|^2_{L^\infty(\hat{B})})\frac{\kappa^2}{\nu^2}|B|\}(1 - e^{-\varepsilon c_1 t})$$
$$+ \frac{8}{(4-\varepsilon)c_1}(|R|^2 + \|\nabla \tilde{\tilde{\theta}}\|^2_{L^\infty(\hat{B})})\frac{1}{4\kappa/\nu - \varepsilon c_1}\|\theta(0)\|^2$$
$$\times (1 - e^{(\varepsilon c_1 - 4\kappa/\nu)t})e^{-\varepsilon c_1 t}.$$

Recalling $\varepsilon < \min(4\kappa/c_1\nu, 4)$ and especially $\varepsilon c_1 - 4\kappa/\nu < 0$, then we get from (17)

$$(18)\quad \| U(t) \|^2 \leq e^{-\varepsilon c_1 t}\| U(0) \|^2 + A_1(1 - e^{-\varepsilon c_1 t}) + A_2\|\theta(0)\|^2 e^{-\varepsilon c_1 t}.$$

Now, we define a mapping τ as follows:

$$(19)\quad \begin{cases} \tau : H = H_\sigma(\Omega(0)) \times L^2(\Omega(0)) \to H, \\ \tau U(0) = U(T) \text{ in } H, \end{cases}$$

where we use the assumption $\Omega(0) = \Omega(T)$. By Proposition 1 we can take initial data U_0 in $H = H_\sigma(\Omega(0)) \times L^2(\Omega(0))$. So, τ can be defined on H. We also see τ is continuous in H. Moreover, by Lemma 4, we infer that $\tau U(0) = U(T)$ is included in a bounded set of $\dot{H}_\sigma^1(\Omega(0)) \times \mathring{W}_2^1(\Omega(0))$, from this fact, it follows that τ is a compact mapping $H \to H$. Thus, we can apply Schauder's fixed point theorem to τ. Indeed, if we choose a constant $r > 0$ such that $2A_1 \leq r^2$ holds, then for any initial value $U(0)$ satisfying $\| U(0) \| \leq r$ we get from (18)

$$(20)\quad \| U(T) \|^2 \leq \frac{r^2}{2} + \frac{r^2}{2}(1 + 2A_2)e^{-\varepsilon c_1 T}.$$

Here we put $T_0 = (\varepsilon c_1)^{-1}\log(1 + 2A_2)$ and assume $T_0 \leqq T$, then $\| U(t) \|^2$

$\leqq r^2$ holds and τ maps the closed ball $B_r \equiv \{\Phi \in H = H_\sigma(\Omega(0)) \times L^2(\Omega(0))\ ;\ \|\Phi\|_H \leq r\}$ into itself. Consequently, by Schauder's fixed point theorem, there exists $V_0 \in H_\sigma(\Omega(0)) \times L^2(\Omega(0))$ such that $\tau V_o = V_o$. Hence we have shown (i).

Next we prove (ii). Let U_Π be the periodic strong solution obtained in (i) and U_1 be any periodic strong solution of (AHC). Put $W = U_\Pi - U_1$, then we have

$$(21)\quad \frac{1}{2}\frac{d}{dt}\|W(t)\|^2 + 2\varphi^t(W(t)) \leq c_5\varphi^t(W(t))\cdot\varphi^t(U_\Pi(t))^{1/2} + c_6N(t)\varphi^t(W(t)) \text{ for a.e. } t \in [0, T],$$

where c_5 and c_6 are positive constants independent of t ; $N(t) = \|\nabla b(t)\| + \|\nabla \bar{\bar{\theta}}(t)\| + |R|$. Recall Remark 2 and the assumptions on b and $\bar{\theta}$, then, by virtue of (i) and (ii) of Lemma 4, $\varphi^t(U_\Pi(t))$ is so small that for any $t \in [0, T], 2 - c_5\varphi^t(U_\Pi(t))^{1/2} - c_6N(t) > 0$ holds.
Hence, by using an elementary argument, we proved the uniqueness of the periodic strong solution.

Proof of Theorem 2. The claim (i) is an immediate consequence of Proposition 1. We can prove (ii) in the same way as in the proof of (ii) of Theorem 1. So, we omit the details.

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