16. *Some Generalizations of the Unicity Theorem of Nevanlinna*

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(Communicated by Kiyosi ITÔ, M. J. A., March 12, 1993)

1. Introduction. Let $f(z)$ be a transcendental meromorphic function in $|z| < \infty$ and let $S(f)$ be the set of meromorphic functions $a(z)$ in $|z| < \infty$ which satisfy

$$T(r, a) = o(T(r, f)) \quad (r \to \infty).$$

We consider $\bar{C} = C \cup \{\infty\}$ to be a subset of $S(f)$. We put for $a \in S(f)$

$$E(f = a) = \{z : f(z) - a(z) = 0\}.$$

More than sixty years ago, R. Nevanlinna proved the following theorem, which is called the Unicity Theorem.

Theorem A. *Let f_1 and f_2 be transcendental meromorphic functions in $|z| < \infty$. If for five distinct values $a_1, \ldots, a_5$ of $\bar{C}$*

$$E(f_1 = a_j) = E\{f_2 = a_j) \quad (j = 1, \ldots, 5),$$

then $f_1 = f_2$ ([2], p. 109, see also [1], p. 48).

The following theorem was used to prove Theorem A in [2].

Theorem B. *For any $q (\geq 3)$ distinct values $a_1, \ldots, a_q$ of $\bar{C}$,*

$$(1) \qquad (q-2)T(r, f) < \sum_{j=1}^{q} \bar{N}(r, a_j) + S(r, f)$$

([2], p. 70).

The functions $f_1(z) = e^z$, $f_2(z) = e^{-z}$, with $a_1 = 0$, $a_2 = 1$, $a_3 = -1$ and $a_4 = \infty$ show that Theorem A is best ([2], p. 111).

It is an open problem to generalize Theorem A to the case when $a_1, \ldots, a_5$ belong to $S(f)$ ([3]). This is neither trivial nor easy since we do not have an inequality corresponding to (1) for $a_1, \ldots, a_q$ of $S(f)$ except when $q = 3$. When $q = 3$, we have the following theorem.

Theorem C. *Suppose that a_1, a_2 and a_3 are distinct in $S(f)$. Then we have*

$$(1 + o(1))T(r, f) < \sum_{j=1}^{3} \bar{N}(r, 1/(f - a_j)) + S(r, f)$$

as $r \to \infty$ (see [1], p. 47).

It is a very interesting open problem whether (1) holds for distinct $a_1, \ldots, a_q$ in $S(f)$ ([1], p. 47; cf. [4], Satz 1).

The purpose of this paper is to give some generalizations of Theorem A by making use of Theorem C. We use the standard notation of the Navanlinna theory of meromorphic functions ([1], [2]) and we use ${}_nC_k = n!/(n-k)!k!$ as the binomial coefficient.

2. Lemmas. We shall give some lemmas in this section. Let f be a transcendental meromorphic function in $|z| < \infty$.

Lemma 1. *If* $a_1,\dots,a_7$ *are distinct elements of* $S(f)$, *then*

$$\left(\frac{7}{3}+o(1)\right)T(r,f)<\sum_{j=1}^{7}\bar{N}(r,1/(f-a_j))+S(r,f)$$

as $r\to\infty$.

Proof. For any distinct integers s, t, u such that $1\le s, t, u\le 7$, we have from Theorem C

$$(1+o(1))T(r,f)<\bar{N}(r,1/(f-a_s))+\bar{N}(r,1/(f-a_t))+\bar{N}(r,1/(f-a_u))+S(r,f)$$

as $r\to\infty$. Since there are ${}_7C_3$ different combinations when we choose three elements from $a_1,\dots,a_7$, we obtain

$${}_7C_3(1+o(1))T(r,f)<{}_6C_2\sum_{j=1}^{7}\bar{N}(r,1/(f-a_j))+S(r,f)$$

as $r\to\infty$, which reduces to the inequality to be proved.

Lemma 2. *If* $a_1,\dots,a_6$ *are distinct elements of* $S(f)$, *then*

$$\left(\frac{5}{2}+o(1)\right)T(r,f)<\sum_{j=1}^{5}\bar{N}(r,1/(f-a_j))+\frac{5}{2}\bar{N}(r,1/(f-a_6))+S(r,f)$$

as $r\to\infty$.

Proof. For any distinct integers p, q such that $1\le p, q\le 5$, we have from Theorem C

$$(1+o(1))T(r,f)<\bar{N}(r,1/(f-a_p))+\bar{N}(r,1/(f-a_q))+\bar{N}(r,1/(f-a_6))+S(r,f)$$

as $r\to\infty$. Since there are ${}_5C_2$ different combinations when we choose two elements from $a_1,\dots,a_5$, we obtain

$${}_5C_2(1+o(1))T(r,f)<{}_4C_1\sum_{j=1}^{5}\bar{N}(r,1/(f-a_j))+{}_5C_2\bar{N}(r,1/(f-a_6))+S(r,f)$$

as $r\to\infty$, which reduces to the inequality to be proved.

Lemma 3. *If* $a_1,\dots,a_5$ *are distinct elements of* $S(f)$, *then*

$$(3+o(1))T(r,f)<\sum_{j=1}^{3}\bar{N}(r,1/(f-a_j))+3\{\bar{N}(r,1/(f-a_4))+\bar{N}(r,1/(f-a_5))\}+S(r,f)$$

as $r\to\infty$.

Proof. By Theorem C, we have for $j=1,2,3$

$$(1+o(1))T(r,f)<\bar{N}(r,1/(f-a_j))+\bar{N}(r,1/(f-a_4))+\bar{N}(r,1/(f-a_5))+S(r,f)$$

as $r\to\infty$. Adding these inequalities for $j=1$, 2 and 3, we easily obtain our lemma.

3. Theorems. Let f_1 and f_2 be transcendental meromorphic functions in $|z|<\infty$.

Theorem 1. *If for seven distinct elements* $a_1,\dots,a_7$ *which belong to* $S(f_1)\cap S(f_2)$

$$E(f_1=a_j)=E(f_2=a_j)\quad(j=1,\dots,7),$$

then $f_1=f_2$.

Proof. We suppose that f_1 and f_2 are not identical. We have

$$(2)\quad \left(\frac{7}{3}+o(1)\right)T(r,f_k)<\sum_{j=1}^{7}\bar{N}(r,1/(f_k-a_j))+S(r,f_k)\quad (k=1,2)$$

as $r\to\infty$ by Lemma 1. We write

$$N_j(r)=\bar{N}(r,1/(f_1-a_j))=\bar{N}(r,1/(f_2-a_j))\quad (j=1,\dots,7).$$

We then have from (2) as $r\to\infty$

$$(3)\quad \left(\frac{7}{3}+o(1)\right)\{T(r,f_1)+T(r,f_2)\}<2\sum_{j=1}^{7}N_j(r)+S(r,f_1)+S(r,f_2)$$
$$<2\{T(r,f_1)+T(r,f_2)\}+S(r,f_1)+S(r,f_2)$$

since

$$\sum_{j=1}^{7}N_j(r)\le\bar{N}(r,1/(f_1-f_2)\le T(r,f_1-f_2)+O(1)$$
$$\le T(r,f_1)+T(r,f_2)+O(1).$$

Thus we have

$$\left(\frac{1}{3}+o(1)\right)\{T(r,f_1)+T(r,f_2)\}=S(r,f_1)+S(r,f_2)$$

as $r\to\infty$, which is impossible. f_1 and f_2 must be identical.

Theorem 2. *If there are five distinct elements $a_1,\dots,a_5$ in $S(f_1)\cap S(f_2)$, b_1 in $S(f_1)$ and b_2 in $S(f_2)$ such that b_1 and b_2 are different from $a_1,\dots,a_5$ and such that*

(i) $E(f_1=a_j)=E(f_2=a_j)\quad (j=1,\dots,5)$

(ii) $\limsup_{r\to\infty}\dfrac{\bar{N}(r,1/(f_k-b_k))}{T(r,f_k)}=\delta_k<\dfrac{1}{5}\quad (k=1,2),$

then $f_1=f_2$.

Proof. Suppose that f_1 and f_2 are not identical. We have for $k=1,2$

$$(4)\quad \left(\frac{5}{2}+o(1)\right)T(r,f_k)<\sum_{j=1}^{5}\bar{N}(r,1/(f_k-a_j))+\frac{5}{2}\bar{N}(r,1/(f_k-b_k))$$
$$+S(r,f_k)$$

as $r\to\infty$ by Lemma 2. If we write

$$N_j(r)=\bar{N}(r,1/(f_1-a_j))=\bar{N}(r,1/(f_2-a_j))\quad (j=1,\dots,5),$$

we have from (4) as $r\to\infty$

$$\left(\frac{5}{2}+o(1)\right)\{T(r,f_1)+T(r,f_2)\}$$
$$<2\sum_{j=1}^{5}N_j(r)+\frac{5}{2}\sum_{k=1}^{2}\bar{N}(r,1/(f_k-b_k))+S(r,f_1)+S(r,f_2)$$
$$<\left(2+\frac{5}{2}\delta\right)\{T(r,f_1)+T(r,f_2)\}+S(r,f_1)+S(r,f_2)$$

by the hypothesis (ii), where δ is any number satisfying

$$\max(\delta_1,\delta_2)<\delta<\frac{1}{5}.$$

We also used the inequality

$$\sum_{j=1}^{5}N_j(r)\le\bar{N}(r,1/(f_1-f_2))\le T(r,f_1-f_2)+O(1)$$
$$\le T(r,f_1)+T(r,f_2)+O(1).$$

Thus we have

$$(1-5\delta+o(1))\{T(r,f_1)+T(r,f_2)\}=S(r,f_1)+S(r,f_1)$$

as $r\to\infty$, which is impossible as $1-5\delta>0$. f_1 and f_2 must be identical.

Corollary 1. *If f_1 and f_2 are entire and if there are five distinct elements $a_1,\dots,a_5$ in $S(f_1)\cap S(f_2)-\{\infty\}$ such that*

$$E(f_1=a_j)=E(f_2=a_j)\quad (j=1,\dots,5),$$

then $f_1=f_2$.

Theorem 3. *If there are three distinct elements a_1, a_2 and a_3 in $S(f_1)\cap S(f_2)$, two distinct elements b_1 and c_1 in $S(f_1)$, two distinct elements b_2 and c_2 in $S(f_2)$ such that b_1, c_1, b_2 and c_2 are different from a_1, a_2 and a_3 and such that*

(i) $E(f_1=a_j)=E(f_2=a_j)\quad (j=1,2,3)$

(ii) $\limsup_{r\to\infty}\dfrac{\bar N(r,1/(f_k-b_k))+\bar N(r,1/(f_k-c_k))}{T(r,f_k)}=\delta_k<\dfrac{1}{3}\quad (k=1,2),$

then $f_1=f_2$.

Proof. We suppose that f_1 and f_2 are not identical. We have for $k=1$, 2 as $r\to\infty$

$$(3+o(1))T(r,f_k)<\sum_{j=1}^{3}\bar N(r,1/(f_k-a_j))+3\bar N(r,1/(f_k-b_k))$$
$$+3\bar N(r,1/(f_k-c_k))+S(r,f_k)$$

by Lemma 3. If we write

$$N_j(r)=\bar N(r,1/(f_1-a_j))=\bar N(r,1/(f_2-a_j))\quad (j=1,2,3),$$

as in the proof of Theorem 2 we have as $r\to\infty$

$$(3+o(1))\{T(r,f_1)+T(r,f_2)\}$$
$$<2\sum_{j=1}^{3}N_j(r)+3\sum_{k=1}^{2}\{\bar N(r,1/(f_k-b_k))+\bar N(r,1/(f_k-c_k))\}$$
$$+S(r,f_1)+S(r,f_2)$$
$$<(2+3\delta)\{T(r,f_1)+T(r,f_2)\}+S(r,f_1)+S(r,f_2)$$

by the hypothesis (ii), where δ is any number satisfying

$$\max(\delta_1,\delta_2)<\delta<\frac{1}{3}.$$

Thus we have

$$(1-3\delta+o(1))\{T(r,f_1)+T(r,f_2)\}=S(r,f_1)+S(r,f_2)$$

as $r\to\infty$, which is impossible since $1-3\delta>0$. f_1 and f_2 must be identical.

Corollary 2. *In Theorem 3, if b_k and c_k are Picard exceptional values for $f_k(k=1,2)$, we have $f_1=f_2$.*

This is because the hypothesis (ii) is evidently satisfied in this case.

Remark. For meromorphic functions $f_1(z)$ and $f_2(z)$ in $|z|<1$ which satisfy

$$\limsup_{r\to\infty}\frac{T(r,f_k)}{\log 1/(1-r)}=\infty\quad (k=1,2),$$

similar results to Theorems 1, 2 and 3 remain valid (cf. [1], p. 49).

References

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