15. On Non-starlikeness of Teichmüller Spaces

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Abstract: Recently, Krushkal showed that the universal Teichmüller space is not starlike. In this note, we shall extend his result to Teichmüller spaces of Fuchsian groups whose quotient surfaces have an arbitrarily large disc. Especially, Teichmüller spaces of Fuchsian groups of the second kind are not starlike.

Let H and H^* be the upper half plane and the lower half plane, respectively. The hyperbolic distance between z and w in H is denoted by $d(z, w)$.

Let Γ be a Fuchsian group acting on H, i.e., a discrete subgroup of $PSL(2,\boldsymbol{R})$. The trivial group is denoted by 1. We say that a set X is precisely invariant under the trivial group 1 in Γ if $\gamma(X) \cap X = \phi$ for all $\gamma \in \Gamma - \{id.\}$.

Let $Q(\Gamma)$ be the complex Banach space of bounded holomorphic quadratic differentials for Γ on H^*. Namely, $Q(\Gamma)$ is the set of holomorphic functions φ on H^*, with norm

$$\| \varphi \| = \sup_{z \in H^*} | 4y^2 \varphi(z) | < \infty \ (z = x + iy),$$

and satisfying the functional equation

$$\varphi(\gamma(z))\gamma'(z)^2 = \varphi(z), \ \gamma \in \Gamma.$$

For each $\varphi \in Q(\Gamma)$, there exists a locally schicht holomorphic function W_φ on H^* such that the Schwarzian derivative $\{W_\varphi, z\}$ is equal to $\varphi(z)$. The Teichmüller space $T(\Gamma)$ of Γ is the set of $\varphi \in Q(\Gamma)$ such that W_φ admits a quasiconformal extention to the Riemann sphere $\hat{C}$. Moreover, we denote by $S(\Gamma)$ the set of those $\varphi \in Q(\Gamma)$ for which W_φ are schlicht.

A subset M of $Q(\Gamma)$ is said to be starlike with respect to $\varphi \in M$ if for every $\psi \in M$,

$$\{(1 - t)\varphi + t\psi \, ; o \leq t \leq 1\} \subset M.$$

Our main theorem is:

Theorem 1. *Let Γ be a Fuchsian group acting on H. Suppose that H/Γ has an arbitrarily large disc, i.e., for any positive number r, there exists a hyperbolic disc of radius r which is precisely invariant under the trivial group in Γ. Then $T(\Gamma)$ is not starlike with respect to any point of it.*

Corollary. *Teichmüller spaces of Fuchsian groups of the second kind are not starlike with respect to any of their points.*

We need the following proposition which was essentially proved in [5].

Proposition. *Let $\Gamma_n (n \in N)$ be Fuchsian groups acting on H. Suppose that there is a sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers such that $\Delta(r_n) = \{z \in H \, ; d(z, i) < r_n\}$ is precisely invariant under the trivial group in Γ_n for each n and*

$r_n \to \infty$ as $n \to \infty$. Then for any $\phi \in T(1)$, there is a sequence $\{\phi_n\}_{n=1}^{\infty}$ such that $\phi_n \in T(\Gamma_n)$ and $\{\phi_n\}_{n=1}^{\infty}$ converges to ϕ in compact-open topology.

Proof of Theorem 1. By the assumption, for each $n \in N$, there is an element $\alpha_n \in PSL(2, R)$ and a positive number r_n such that $\Delta(r_n)$ is precisely invariant under the trivial group in $\Gamma_n = \alpha_n^{-1} \Gamma \alpha_n$ and $r_n \to \infty$ as $n \to \infty$.

Suppose that $T(\Gamma)$ is starlike with respect to some point $\varphi \in T(\Gamma)$. Then, $T(\Gamma_n)$ is starlike with respect to the point $\varphi_n(z) = \varphi(\alpha_n(z))\alpha_n'(z)^2$. Since $S(1)$ is compact in compact-open topology, we may assume that $\{\varphi_n\}_{n=1}^{\infty}$ converges to some $\varphi_0 \in S(1)$ in compact-open topology.

Take an isolated point ϕ of $S(1)$. (The existence of such points is due to Thurston. See [1].) Since there are infinitely many such points, we can choose ϕ so that ϕ is distinct from ϕ_0. And it is easy to see that there is a sequence $\{\phi^{(k)}\}_{k=1}^{\infty}$ in $T(1)$ which converges to ϕ in compact-open topology. From the above proposition, for each $k \in N$, there is a sequence $\{\phi_n^{(k)}\}_{n=1}^{\infty}$ such that $\phi_n^{(k)} \in T(\Gamma_n)$ and $\{\phi_n^{(k)}\}_{n=1}^{\infty}$ converges to $\phi^{(k)}$ in compact-open topology. The starlikeness of $T(\Gamma_n)$ implies that

$$\{(1-t)\varphi_n + t\phi_n^{(k)}\,;\, 0 \le t \le 1\} \subset T(\Gamma_n) \subset S(1).$$

For each fixed k, by taking $n \to \infty$, we have

$$\{(1-t)\varphi_0 + t\phi^{(k)}\,;\, 0 \le t \le 1\} \subset S(1).$$

Then by taking $k \to \infty$,

$$\{(1-t)\varphi_0 + t\phi\,;\, 0 \le t \le 1\} \subset S(1).$$

This contradicts the fact that ϕ is an isolated point of $S(1)$.

Remark. *For an arbitrarily large r, there exists a Riemann surface of finite analitic type which contains a hyperbolic disc of radius r. Hence, the proof also shows that there are infinitely many finite-dimensional Teichmüller spaces which are not starlike with respect to the origin.*

By the very similar method, we can obtain the following:

Theorem 2. *Let $\Gamma_n (n \in N)$ be Fuchsian groups acting on H. Suppose that for each $n \in N$, Γ_n contains a hyperbolic element γ_n and as $n \to \infty$ $trace^2(\gamma_n) \to 4$. Then there exists $n \in N$ such that $T(\Gamma_n)$ is not convex.*

Corollary. *Let Γ be an arbitrary Fuchsian group. Then there exists a Fuchsian group Γ' such that Γ' is a quasiconformal deformation of Γ and $T(\Gamma')$ is not convex.*

Proof of Theorem 2. By the assumption, there exsists $\alpha_n \in PSL(2, R)$ such that $\Gamma_n' = \alpha_n^{-1} \Gamma_n \alpha_n$ contains an element $\gamma_n : z \to r_n z (r_n > 1)$ and $r_n \to 1$.

We consider the subspace Q_0 of $Q(1)$ spanned by the element $e(z) = 1/4z^2$, and set $T_0 = Q_0 \cap T(1)$ and $S_0 = Q_0 \cap S(1)$. Then

$$T_0 = \{\varphi = ce\,;\, c = 2(1-\delta^2), |\delta - 1| < 1\}$$

and $S_0 = \bar{T}_0$. (See [2].)

Hence, there are two points $\phi^{(j)} (j = 1,2)$ in T_0 such that $2^{-1}(\phi^{(1)} + \phi^{(2)}) \notin S_0$. Then by the proof of Theorem A in [4], for each j, there exsists a sequence $\{\phi_n^{(j)}\}$ such that $\phi_n^{(j)} \in T(\Gamma_n')$ and $\{\phi_n^{(j)}\}$ converges to $\phi^{(j)}$ in compact-open topology.

If we suppose that all $T(\Gamma_n')$ are convex, we are led to a contradiction as in the proof of Theorem 1.

Hence for some n, $T(\Gamma_n')$ is not convex, so neither is $T(\Gamma_n)$.

References

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