89. *Q-rationality of Moment Maps*

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The purpose of this note is to announce our recent results (see Theorems A, B and C) associated with the $\boldsymbol{Q}$-rationality of moment maps.

Let X be a compact complex connected manifold carrying a Kähler class κ in $H^2(X, \boldsymbol{Q})$. Then we can choose a very ample line bundle L satisfying $c_1(L) = m\kappa$ for some positive integer m, so that we can regard X as a projective algebraic manifold. Put $n := \dim_{\boldsymbol{C}} X$. Assume further that X admits an effective biregular action of the r-dimensional algebraic torus

$$G = \boldsymbol{G}_{\mathrm{m}}^r = \{(z_1, z_2, \ldots, z_r) \,;\, z_\alpha \in \boldsymbol{C}^* \text{ for all } \alpha\}.$$

Let $\mathfrak{g} = \sum_{\alpha=1}^r \boldsymbol{C}\mathcal{X}_\alpha$ be the Lie algebra of G, where $\mathcal{X}_\alpha := \sqrt{-1}\, z_\alpha \partial/\partial z_\alpha$. For the maximal compact subgroup $G_{\boldsymbol{R}} \cong (S^1)^r$ of G, consider the associated real Lie subalgebra $\mathfrak{g}_{\boldsymbol{R}} = \sum_{\alpha=1}^r \boldsymbol{R}\mathcal{X}_\alpha$ of $\mathfrak{g}$. Moreover, $\mathfrak{g}$ has a natural $\boldsymbol{Q}$-structure by $\mathfrak{g}_{\boldsymbol{Q}} = \sum_{\alpha=1}^r \boldsymbol{Q}\mathcal{X}_\alpha$. Take a $G_{\boldsymbol{R}}$-invariant Kähler form ω on X in the class κ. Now, to each $\mathcal{Y} \in \mathfrak{g}_{\boldsymbol{R}}$, we can uniquely associate a Hamiltonian function $\mu_\omega^{\mathcal{Y}}$ on X such that

$$\bar{\partial}\mu_\omega^{\mathcal{Y}} = i_{\mathcal{Y}}(2\pi\omega),$$

where $\mu_\omega^{\mathcal{Y}}$ is real-valued and is required to satisfy the normalization condition $\int_X \mu_\omega^{\mathcal{Y}}\, \omega^n = 0$. Let $\mu_\omega : X \to \mathfrak{g}_{\boldsymbol{R}}^*$ be the moment map defined by setting

$$\langle \mu_\omega(x), \mathcal{Y} \rangle = \mu_\omega^{\mathcal{Y}}(x), \quad \mathcal{Y} \in \mathfrak{g}_{\boldsymbol{R}},$$

for each $x \in X$. This moment map is intrinsic in the sense that it is free from any ambiguity of translation caused by the choice of a G-linearization (cf. Mumford and Forgarty [6]) of a power of L. Let $\{\mathcal{X}_1^*, \ldots, \mathcal{X}_r^*\}$ be the $\boldsymbol{R}$-basis for $\mathfrak{g}_{\boldsymbol{R}}^*$ dual to $\{\mathcal{X}_1, \ldots, \mathcal{X}_r\}$ for $\mathfrak{g}_{\boldsymbol{R}}$. The $\boldsymbol{R}$-basis $\{\mathcal{X}_1^*, \ldots, \mathcal{X}_r^*\}$ allows us to identify $\mathfrak{g}_{\boldsymbol{R}}^*$ with $\boldsymbol{R}^r$, so that μ_ω is rewritten as follows:

$$\mu_\omega(x) = (\mu_\omega^{\mathcal{X}_1}(x), \mu_\omega^{\mathcal{X}_2}(x), \ldots, \mu_\omega^{\mathcal{X}_r}(x)) \in \boldsymbol{R}^r, \quad x \in X.$$

Note the following standard fact (due to Atiyah [1], Guillemin and Sternberg [4]) that the image $\mu_\omega(X)$ of the moment map μ_ω is the convex hull of the finite set $\mu_\omega(X^G)$ in $\boldsymbol{R}^r$, where X^G denotes the fixed point set of the G-action on X. Note also that $\dim_{\boldsymbol{R}} \mu_\omega(X) = r$. Let $\mathrm{Crt}(\mu_\omega)$ be the set of all critical values for μ_ω. As in the case [4] of a moment map (which differs from our intrinsic μ_ω by a translation) associated with a G-linearization of a power of L, we have the following:

Theorem A. *The finite subset $\mu_\omega(X^G)$ of $\boldsymbol{R}^r$ sits in $\boldsymbol{Q}^r$. Moreover, $\boldsymbol{R}^r$ naturally admits a finite number of real linear subspaces $H_1, H_2, \ldots, H_p$, all defined over $\boldsymbol{Q}$ and not necessarily passing through the origin, such that*

(1) $\mathrm{Crt}(\mu_\omega) \subset \cup_{j=1}^p H_j$;

(2) $H_j \subsetneq \boldsymbol{R}^r$ *for all* j.

We next consider the symplectic reduction. If $t \in \mathrm{Reg}(\mu_\omega) := \mu_\omega(X) \setminus \mathrm{Crt}(\mu_\omega)$, i.e., t is the regular value for μ_ω, then by [2], there exists a Kähler form η_t on the orbifold $M_t := \mu_\omega^{-1}(t)/G_{\boldsymbol{R}}$ such that

$$\iota_t^*(\omega) = p_t^*(\eta_t),$$

where $\iota_t : \mu_\omega^{-1}(t) \hookrightarrow X$ and $p_t : \mu_\omega^{-1}(t) \to \mu_\omega^{-1}(t)/G_{\boldsymbol{R}} = M_t$ are natural maps. Let $[\eta_t]$ be the cohomology class in $H^2(M_t, \boldsymbol{R})$ represented by η_t. We then obtain:

Theorem B. *If* $t \in \mathrm{Reg}(\mu_\omega)$ *is in* $\boldsymbol{Q}^r$, *then* $[\eta_t] \in H^2(M_t, \boldsymbol{Q})$.

We finally give an application of these results. Consider the symmetric $\boldsymbol{C}$-bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \boldsymbol{C}$ defined by

$$B(\mathscr{Y}_1, \mathscr{Y}_2) := \int_X \mu_\omega^{\mathscr{Y}_1} \mu_\omega^{\mathscr{Y}_2}\, \omega^n / n!, \quad \mathscr{Y}_1, \mathscr{Y}_2 \in \mathfrak{g}.$$

Then by [3] (see also [5 ; Corollary 5.2]), this bilinear form depends only on the class κ and is independent of the choice of ω in κ. Our application is the following:

Theorem C. *The bilinear form B is defined over* $\boldsymbol{Q}$, *i.e.*, $B(\mathscr{X}_\alpha, \mathscr{X}_\beta) \in \boldsymbol{Q}$ *for all* α *and* β.

We here explain how Theorem C is obtained from Theorems A and B. Write $\mathrm{Reg}(\mu_\omega)$ as a disjoint union $\cup_{l=1}^{\nu} C_l$ of its connected components. Since the map $\mu_\omega : X \to \boldsymbol{R}^r$ defines a locally $G_{\boldsymbol{R}}$-equivariantly trivial family of compact differentiable manifolds over the set $\mathrm{Reg}(\mu_\omega)$ of regular values, we have a natural identification of $H^2(M_t, K)$ with $H^2(M_{t'}, K)$ for $K = \boldsymbol{R}$ or $\boldsymbol{Q}$, if t and t' are in the same connected component C_l of $\mathrm{Reg}(\mu_\omega)$. Hence, we put $\Lambda_l(K) := H^2(M_t, K) = H^2(M_{t'}, K)$ for $t, t' \in C_l$. In view of this identification, a result of Duistermaat and Heckman [2] states the following:

Fact. *For each* l, *there exist elements* $d_1^{(l)}, d_2^{(l)}, \ldots, d_r^{(l)}$ *in* $\Lambda_l(\boldsymbol{Q})$ *such that*

$$(1) \qquad [\eta_{t'}] = [\eta_t] + \Sigma_{\alpha=1}^{r} (t'_\alpha - t_\alpha) d_\alpha^{(l)},$$

for all $t = (t_1, t_2, \ldots, t_r)$ *and* $t' = (t'_1, t'_2, \ldots, t'_r)$ *in* C_l. *Therefore, the pushforward* $(\mu_\omega)_*(\omega^n/n!)$ *of the measure* $\omega^n/n!$ *by* μ_ω *is a piecewise polynomial measure on* $\boldsymbol{R}^r$ *characterized by*

$$(2) \qquad \{(\mu_\omega)_*(\omega^n/n!)\}(t) = \left\{\int_{M_t} \eta_t^{n-r}/(n-r)!\right\} dt,$$

where $dt := dt_1 \wedge dt_2 \wedge \cdots \wedge dt_r$ *denotes the standard Lebesgue measure on the vector space* $\mathfrak{g}_{\boldsymbol{R}}^* \cong \boldsymbol{R}^r$ *in terms of the real basis* $\mathscr{X}_1^*, \mathscr{X}_2^*, \ldots, \mathscr{X}_r^*$.

For each l, we choose a point $e^{(l)} = (e_1^{(l)}, e_2^{(l)}, \ldots, e_r^{(l)})$ in $C_l \cap \boldsymbol{Q}^r$. In view of $\mu_\omega^* t_\alpha = \mu_\omega^{\mathscr{X}_\alpha}$ and $\mu_\omega^* t_\beta = \mu_\omega^{\mathscr{X}_\beta}$, it then follows from (1) and (2) that

$$B(\mathscr{X}_\alpha, \mathscr{X}_\beta) = \int_X \mu_\omega^{\mathscr{X}_\alpha} \mu_\omega^{\mathscr{X}_\beta}\, \omega^n/n! = \int_{\mu_\omega(X)} t_\alpha t_\beta (\mu_\omega)_*(\omega^n/n!)$$

$$= \sum_{l=1}^{\nu} \int_{C_l} t_\alpha t_\beta \left\{\int_{M_{e^{(l)}}} (\eta_{e^{(l)}} + \Sigma_{\alpha=1}^{r} (t_\alpha - e_\alpha^{(l)}) d_\alpha^{(l)})^{n-r}/(n-r)!\right\} dt.$$

This together with Theorems A and B completes the proof of Theorem C.

References

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