46. *Some New Examples of Eigenmaps from S^m into S^n*

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1. Introduction. Recently, H. Gauchman and G. Toth introduced a method of constructing new examples of eigenmaps between spheres from known old ones ([1]). As a consequence, they proved the following theorem by applying it to those λ_2-eigenmaps obtained in [2]-[4].

Theorem A. *For $m \geq 5$, there exist full λ_2-eigenmaps*

$$f : S^m \to S^{\frac{m(m+3)}{2}-r},$$

where $r = 1,2,3,4,5,7,11,12,13,16$.

Here a map $f : S^m \to S^n$ is said to be λ_2-eigenmap if all components of f are spherical harmonics of degree 2, and is called full if its image is not contained in any totally hypersphere of S^n.

Theorem A implies, in particular, that full λ_2-eigenmaps $f : S^5 \to S^n$ exist for $n = 4,7,8,9,13,15,16,17,18,19$. However, in their approach the existence of full λ_2-eigenmaps $f : S^5 \to S^n$ is missing for $n = 3,5,6,10,11,12,14$ (for $n = 2$ the non-existence is proved in [1]).

The aim of this note is to show the following theorem which supplements Theorem A.

Theorem B. *Let $k \geq 1$. Then the following hold.*

(i) *There exist full λ_2-eigenmaps $f : S^{2k+1} \to S^l$ for $k^2 + 3k \leq l \leq 2k^2 + 4k + 2$, $l = k^2 + 3k - 2$.*

(ii) *There exist full λ_2-eigenmaps $f : S^{2k+2} \to S^l$ for $k^2 + 5k + 3 \leq l \leq 2k^2 + 6k + 5$, $l = k^2 + 5k - 2 + 2s(k-1)$ $(0 \leq s \leq k+1)$ or $l = k^2 + 5k + 1$.*

Our method of proof is different from that of H. Gauchman and G. Toth and, in fact, makes an essential use of orthogonal multiplications $\boldsymbol{R}^2 \times \boldsymbol{R}^n \to \boldsymbol{R}^r$ in constructing these maps. As a corollary of Theorem B, we obtain for instance

Corollary. There exist full λ_2-eigenmaps $f : S^5 \to S^n$ for $n = 10,11,12,14$.

This corollary combined with a result in H. Gauchman and G. Toth then implies that Theorem A is true for r such that $1 \leq r \leq 13$ or $r = 16$.

2. Existence of orthogonal multiplication for $m = 2$. An orthogonal multiplication $F : \boldsymbol{R}^m \times \boldsymbol{R}^n \to \boldsymbol{R}^r$ is by definition a bilinear map such that $\| F(x, y) \| = \| x \| \cdot \| y \|$, where $\| \cdot \|$ denotes the Euclidean norm. F is said to be full if the image of F spans $\boldsymbol{R}^r$.

It is well-known that if $F : \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}^r$ is an orthogonal multiplication, then the Hopf map defined by

$$f_F(x, y) := (\| x \|^2 - \| y \|^2, 2F(x, y)), \quad x, y \in \boldsymbol{R}^n$$

gives rise to a λ_2-eigenmap from S^{2n-1} into S^r. Note that from its definition the Hopf map is defined only on odd dimensional spheres. However, for even dimensional spheres S^k we can construct λ_2-eigenmaps $f : S^k \to S^l$ by orthogonal multiplications as follows.

Given two λ_2-eigenmaps $g : S^{m-1} \to S^{p-1}$, $h : S^{n-1} \to S^{q-1}$ and an orthogonal multiplication $F : \boldsymbol{R}^m \times \boldsymbol{R}^n \to \boldsymbol{R}^r$, we define

(1) $$f(x, y) := (g(x), h(y), \sqrt{2}F(x, y)),$$

where $x \in \boldsymbol{R}^m$, $y \in \boldsymbol{R}^n$. Then it follows from $\| g(x) \| = \| x \|^2$, $\| h(y) \| = \| y \|^2$ and $\| F(x, y) \| = \| x \| \cdot \| y \|$ that

$$\| f(x, y) \|^2 = (\| x \|^2 + \| y \|^2)^2,$$

which implies that f gives rise to a λ_2-eigenmap $f : S^{m+n-1} \to S^{p+q+r-1}$. Moreover, if g and h are full λ_2-eigenmaps and F is a full orthogonal multiplication, then f defines a full λ_2-eigenmap.

In this section, we shall prove the existence of orthogonal multiplications $F : \boldsymbol{R}^m \times \boldsymbol{R}^n \to \boldsymbol{R}^r$ when $m = 2$.

We assume $m \leq n$. Since $F : \boldsymbol{R}^m \times \boldsymbol{R}^n \to \boldsymbol{R}^r$ is a bilinear map, we may write

$$F(x, y) = \sum a_{ij} x_i y_j,$$

where $x = (x_1, \ldots, x_m) \in \boldsymbol{R}^m$, $y = (y_1, \ldots, y_n) \in \boldsymbol{R}^n$ and $a_{ij} \in \boldsymbol{R}^r$. Since $\| F(x, y) \| = \| x \| \cdot \| y \|$, we have

(2) $$\begin{cases} \langle a_{ik}, a_{il} \rangle = \delta_{kl} \\ \langle a_{ik}, a_{jk} \rangle = \delta_{ij} \\ \langle a_{ik}, a_{jl} \rangle + \langle a_{il}, a_{jk} \rangle = 0 \quad (i \neq j,\ k \neq l), \end{cases}$$

which imply that $\{a_{1k}\}_{k=1}^n$ is an orthogonal system in $\boldsymbol{R}^r$, and hence $n \leq r$. Moreover, since F is full, we have $n \leq r \leq mn$.

Following the method due to M. Parker [2], who employed it in the case $m = n$, we now define an $mn \times mn$-matrix $G(F)$ by

$$G(F) := \begin{bmatrix} I_n & A_{12} & \cdots & A_{1m} \\ A_{21} & I_n & \cdots & A_{2m} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & I_n \end{bmatrix},$$

where I_n denotes the $n \times n$ identity matrix and A_{ij} the $n \times n$-matrix whose entries are

$$(A_{ij})_{kl} = \langle a_{ik}, a_{jl} \rangle, \quad 1 \leq k, l \leq n.$$

Owing to (2), A_{ij} is a skew-symmetric matrix and $A_{ji} = -A_{ij}$.

Note that the determinant $\det G(F)$ of $G(F)$ coincides with the Gram's determinant with respect to the system of vectors $\{a_{ij}\}$. Hence it holds that rank $G(F) = r$.

We set $m = 2$. Then $G(F)$ is a $2n \times 2n$-matrix given by

$$G(F) = \begin{bmatrix} I_n & -A \\ A & I_n \end{bmatrix}, \quad A = -A_{12}.$$

Proposition C. *A full orthogonal multiplication $F : \boldsymbol{R}^2 \times \boldsymbol{R}^n \to \boldsymbol{R}^r$ exists if and only if r is even.*

Proof. First, we prove that rank $G(F)(= r)$ is even whenever a full orthogonal multiplication exists. Recall that the characteristic polynomial of $G(F)$ is

$$\det(G(F) - \mu I_{2n}) = \det \begin{bmatrix} (1-\mu)I_n & -A \\ A & (1-\mu)I_n \end{bmatrix}.$$

Noting the formula

$$\det \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = |\det[A + \sqrt{-1}B]|^2,$$

where A and B are real matrices and $|\cdot|$ denotes absolute value, we have

$$\det(G(F) - \mu I_{2n}) = |\det[(1-\mu)I_n + \sqrt{-1}A]|^2.$$

Since A is skew-symmetric, $\det[(1-\mu)I_n + \sqrt{-1}A] \in \boldsymbol{R}$, and therefore

(3) $$\det(G(F) - \mu I_{2n}) = \{\det[(1-\mu)I_n + \sqrt{-1}A]\}^2.$$

On the other hand, there exists a full orthogonal multiplication $F : \boldsymbol{R}^2 \times \boldsymbol{R}^n \to \boldsymbol{R}^{2n}$ exists and is defined by

(4) $$F(x, y) = (x_1y_1, x_2y_1, x_1y_2, x_2y_2, \ldots, x_1y_n, x_2y_n),$$

where $x = (x_1, x_2) \in \boldsymbol{R}^2$, $y = (y_1, \ldots, y_n) \in \boldsymbol{R}^n$, which realizes the case $r = 2n$. Hence it follows from (3) that r is even.

Next, note that two vectors $(x_1y_i, x_2y_i, x_1y_j, x_2y_j)$ and $(x_1y_i + x_2y_j, x_1y_j - x_2y_i)$ have the same norm. Then it is easy to see that an orthogonal multiplication $F^{(1)} : \boldsymbol{R}^2 \times \boldsymbol{R}^n \to \boldsymbol{R}^{2n-2}$ is obtained from F in (4) by

$$F^{(1)}(x, y) = (x_1y_1 + x_2y_2, x_1y_2 - x_2y_1, x_1y_3, x_2y_3, \ldots, x_1y_n, x_2y_n),$$

since $\| F^{(1)}(x, y) \| = \| F(x, y) \|$. Similarly, $F^{(2)} : \boldsymbol{R}^2 \times \boldsymbol{R}^n \to \boldsymbol{R}^{2n-4}$ from $F^{(1)}$ by

$$F^{(2)}(x, y) = (x_1y_1 + x_2y_2, x_1y_2 - x_2y_1, x_1y_3 + x_2y_4, x_1y_4 - x_2y_3, \ldots, x_1y_n, x_2y_n),$$

which is an orthogonal multiplication satisfying $\| F^{(2)}(x, y) \| = \| F^{(1)}(x, y) \| = \| F(x, y) \|$. Repeating this process, we can inductively define orthogonal multiplications

$$\tilde{F} : \boldsymbol{R}^2 \times \boldsymbol{R}^n \to \boldsymbol{R}^{2(n-s)}$$

for $0 \leq s \leq \left[\frac{n}{2}\right]$, where $[\alpha]$ denotes the maximal integer such that $[\alpha] \leq \alpha$. Hence the proposition follows.

3. Proof of Theorem B. We first note that it is known by H. Gauchman and G. Toth ([1]) that there exist full λ_2-eigenmaps

$h : S^3 \to S^q$ for $q = 2,4,5,6,7,8$, and

$h : S^4 \to S^q$ for $q = 4,7,9,10,11,12,13$.

We are going to prove the odd dimensional case (i). Even dimensional case (ii) can be proved by the same argument.

From Proposition C, there exist full orthogonal multiplications $F : \boldsymbol{R}^2 \times \boldsymbol{R}^{2k+2} \to \boldsymbol{R}^r$ for $r = 2k + 2 + 2s$ $(0 \leq s \leq k+1)$. We assume that full λ_2-eigenmaps $h : S^{2k+1} \to S^q$ exist for all q satisfying $q_k \leq q \leq \tilde{q}_k$ ($q_k, \tilde{q}_k \in \boldsymbol{N}$ and $q_k < \tilde{q}_k$). Set

(5) $$q_{k+1} := q_k + 2k + 4, \quad \tilde{q}_{k+1} := \tilde{q}_k + 4k + 6.$$

Then we can construct full λ_2-eigenmaps

$$f : S^{2k+3} \to S^l$$

for $q_{k+1} \leq l \leq \tilde{q}_{k+1}$ in the following fashion.

Let $g(x_1, x_2) = (|x_1|^2 - |x_2|^2, 2x_1x_2)$, and define

$$f(x, y) := (g(x), h(y), \sqrt{2}F(x, y)),$$

h and F being as above. Then $f : S^{2k+3} \to S^{q+r+2}$ gives rise to a full λ_2-eigenmap. Since $q_k \leq l \leq \tilde{q}_k$ and $r = 2k + 2 + 2s$ $(0 \leq s \leq k + 1)$, $q + r + 2$ takes any integer satisfying

$$q_k + 2k + 4 \leq q + r + 2 \leq \tilde{q}_k + 4k + 6.$$

Now, recall that when $k = 1$, the above examples due to H. Gauchman and G. Toth show that full λ_2-eigenmap $h : S^3 \to S^q$ exists for $4 \leq q \leq 8$. Hence we may take $q_1 = 4$ and $\tilde{q}_1 = 8$. Then it follows from (5) that

$$q_k = k^2 + 3k, \quad \tilde{q}_k = 2k^2 + 4k + 2.$$

Hence we obtain full λ_2-eigenmaps $f : S^{2k+1} \to S^q$ for $k^2 + 3k \leq q \leq 2k^2 + 4k + 2$.

On the other hand, by making use of $h : S^3 \to S^2$, we obtain similarly full λ_2-eigenmaps

$$f : S^5 \to S^{8+2s} \quad \text{for } 0 \leq s \leq 2,$$

from which we can also construct inductively full λ_2-eigenmaps

$$f : S^{2k+1} \to S^{t_k},$$

where $t_k = k^2 + 3k - 2 + 2s(k - 1)$ $(0 \leq s \leq k)$. Note that since $t_k \geq q_k$ if $s \geq 1$, these examples are contained in the previous case when $s \geq 1$.

Consequently, we see that full λ_2-eigenmaps

$$f : S^{2k+1} \to S^l$$

exist for $l = k^2 + 3k - 2$ or $k^2 + 3k \leq l \leq 2k^2 + 4k + 2$.

Remark. By the same argument we can construct more examples of full λ_2-eigenmaps. For example, it is known by the result due to M. Parker that full orthogonal multiplications $F : \boldsymbol{R}^3 \times \boldsymbol{R}^4 \to \boldsymbol{R}^r$ exist for $r = 4,7,8,10,11,12$. By making use of this, we can construct full λ_2-eigenmaps

$$f : S^6 \to S^l$$

for $l = 12,14$ and $15 \leq l \leq 26$.

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